POLYNOMIAL SOLUTIONS OF THE PLANE PROBLEM
IN THE THEORY OF ELECTROELASTICITY

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§1. General solutions to the plane problem of the theory of electroelasticity were derived in [1]. The unknown functions were expressed in terms of arbitrary functions of a complex variable of special form. These functions had to satisfy certain boundary conditions, and great difficulties arose in connection with this.

For a number of important particular cases we may find a solution in the form of complete polynomials. Let us consider the cross section of a rod having the form of a long rectangular strip; in the plane of this strip the coordinate axes $x_1$ and $x_2$ lie parallel to the sides of the cross section, so that $0 \leq x_1 \leq l$, $-c \leq x_2 \leq c$. We shall consider that the conditions of the plane problem derived in [1] are still satisfied.

In order to study the direct piezoeffect, we shall assume that the piezoelectric crystal lies in an isolated medium not containing any conducting solids, and that its surfaces are not electrically connected. This means that over the whole contour $D_n = 0$, where $n$ is the external normal to the contour of the cross section.

By virtue of the homogeneity of the crystal we may conclude that in the absence of space charge, the vector $D_i = 0$ over the whole volume as well. Hence the function $\phi$ related to the vector $D_i$ by the equations of [1] may be taken as identically equal to zero, and the problem reduces to one of seeking the function $\psi$ from the differential equation

$$L_\lambda \psi = 0.$$  (1.1)

The second equation of the general system of differential equations

$$L_\lambda \psi = 0$$  (1.2)

constitutes the condition imposed upon the material constants of the crystal in which the electric field exists.

As in the classical theory of elasticity [2, 3] we shall seek the stress function in the form

$$\psi = \sum_{i=1}^n \sum_{k=0}^i A_{i,k+1} x_1^i x_2^k.$$  (1.3)

The polynomials of second and third degree are solutions of Eqs. (1.1) for any values of the coefficients $A_{1,k+1}$. In the polynomials of higher degree we may specify four coefficients quite arbitrarily; the remaining constants have to be determined as functions of these four.

Let us consider the polynomial of the second degree

$$\psi = A_{21} x_1^2 + A_{22} x_1 x_2 + A_{23} x_2^2,$$  (1.4)

in which the stresses will be $\sigma_{11} = 2A_{23}$, $\sigma_{22} = 2A_{21}$, $\sigma_{12} = -A_{22}$. If all the $A_{2,k+1} \neq 0$, the stressed state will constitute the combined action of a uniformly-distributed tension (compression) in two mutually-perpendicular directions and pure shear. Clearly the function (1.4) will satisfy Eq. (1.2) identically. On the basis of the results presented in [1] we may find the components of the electric-field vector and the relative deformations.


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By integration we may now find the electric-field potential $U$ and the displacement vector of the points $u_i$ ($i = 1, 2$). From the known quantities (1.5) we determine $D_3$, $\sigma_{33}$, $\sigma_{23}$, $\sigma_{13}$.

Let us consider the stress function in the form of a polynomial of the third degree

$$
\psi = A_{34} x_1^3 + A_{33} x_2^3 + A_{32} x_1 x_2^2 + A_{31} x_1^2 x_2 + A_{30} x_1^3 + A_{23} x_1^2 x_2 + A_{22} x_1 x_2^2 + A_{21} x_2^3 + A_{13} x_1^3 + A_{12} x_1^2 x_2 + A_{11} x_1 x_2^2 + A_{03} x_2^3 + A_{02} x_1^2 x_2 + A_{01} x_1 x_2^2 + A_{00} x_2^3
$$

which satisfies Eq. (1.1) for any values of the coefficients. Equation (1.2) gives the condition for the development of an electric field

$$
-3b_{24} A_{31} + (b_{14} + b_{23}) A_{32} - (b_{15} + b_{25}) A_{33} + 3b_{14} A_{34} = 0.
$$

If we put all the coefficients of the polynomial equal to zero except for $A_{34}$, we obtain pure bending by virtue of the normal stresses $\sigma_{11} = 6A_{34}x_1$, distributed in accordance with a linear law on the area perpendicular to the $x_1$ axis. In this case an electric signal arises subject to the condition $b_{13} = 0$.

The electric-field vector equals $E_1 = 0$; $E_2 = 6b_{34} A_{34} x_2$. From this we see that the lines $x_2 = \text{const}$ are equipotentials. The relative deformations have the form

$$
\xi_{11} = 6b_{33} A_{34} x_2^3; \quad \xi_{12} = 6b_{34} A_{34} x_2^2.
$$

The displacement vector is determined by the expressions

$$
\begin{align*}
U_1 &= 6b_{33} A_{34} x_2^3 + 3b_{43} A_{34} x_2^2 - \omega x_2 + u_{10}; \\
u_1 &= -3b_{34} A_{34} x_1^3 + 3b_{44} A_{34} x_1^2 - \omega x_1 + u_{20}.
\end{align*}
$$

Here $\omega$, $u_{10}$, $u_{20}$ are constants which have to be determined from the fixing condition.

The equation of the bent axis of a beam with fixed ends $x_1 = 0$, $x_2 = l$ takes the form $\eta = 3b_{34} A_{34} (lx_1 - x_1^3)$, while the sag amounts to $\eta_0 = (3/4)b_{34} A_{34} l^2$.

The equation of the bent axis and the sag of a beam fixed in the cantilever manner in the section $x_1 = 0$, $x_2 = 0$ are as follows:

$$
\eta = -3b_{34} A_{34} x_1^3; \quad \eta_0 = -3b_{34} A_{34} l^2.
$$

In the same way we may analyze the case in which only $A_{31}$ \(=\) 0. We then obtain pure bending by virtue of the normal stresses acting on the area perpendicular to the $x_2$ axis.

By way of an example of the use of a polynomial of the fourth degree, let us consider the problem of cantilever bending by a force $P$ applied to the free end $x_1 = l$. On the sides $x_2 = \pm c$ all stresses are absent, i.e.,

$$
\sigma_{31} = \sigma_{32} = 0.
$$

Intersecting the beam by a plane perpendicular to the $x_1$ axis at a distance $x_1$ from the origin of coordinates, considering the equilibrium of the cut-off part we obtain three further conditions for the stresses

$$
\int_0^x \sigma_{11} dx_1 = 0; \quad \int_0^x \sigma_{12} dx_2 = 0; \quad \int_0^x \sigma_{13} dx_2 = 0; \quad \int_0^x \sigma_{14} dx_2 = -P.
$$

We shall seek the solution of Eq. (1.1) in the form of a sum of polynomials of the second, third, and fourth degree

$$
\psi = A_{24} x_1^2 + A_{23} x_1 x_2 + A_{22} x_2^2 + A_{21} x_1^2 + A_{20} x_1^3 + A_{14} x_1^2 + A_{13} x_1 x_2 + A_{12} x_1 x_2 + A_{11} x_2^2 + A_{10} x_1^3 + A_{03} x_2^3 + A_{02} x_1^2 x_2 + A_{01} x_1 x_2^2 + A_{00} x_2^3
$$

Finding the stresses and satisfying the conditions (1.10) we have

$$
A_{41} = A_{42} = A_{43} = A_{44} = A_{40} = 0; \quad A_{22} + 3A_{41} = 0.
$$

The stress functions take the form

$$
\psi = A_{24} x_1^2 + A_{23} x_1 x_2 + A_{22} x_2^2 + A_{21} x_1^2 + A_{20} x_1^3 + A_{14} x_1^2 + A_{13} x_1 x_2 + A_{12} x_1 x_2 + A_{11} x_2^2 + A_{10} x_1^3 + A_{03} x_2^3 + A_{02} x_1^2 x_2 + A_{01} x_1 x_2^2 + A_{00} x_2^3
$$

(1.14)