THEORY OF PLATES SUBJECTED TO FINITE INITIAL DEFORMATIONS

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1. We consider a plate with initial thickness $h_0$ made of an elastic homogeneous isotropic compressible material. Assume that due to a finite uniform deformation along the three axes of a rectangular system of coordinates $x_1, x_2, x_3$, the plate changes from the initial nondeformed and nonstressed state $B_0$ into the state $B$, which is characterized by the coordinates $x, y, z$, equal to

$$x = \lambda_1 x_1; \quad y = \lambda_2 x_2; \quad z = \lambda_3 x_3,$$

where $\lambda_i$ are the parameters of the deformation.

The components of the stress tensor, supporting the body in the state $B$, are obtained from the formulas [5]

$$\tau^{11} = \lambda_1^3 \Phi + \lambda_1^2 (\lambda_2^2 + \lambda_3^2) \Psi + P;$$
$$\tau^{22} = \lambda_2^3 \Phi + \lambda_2^2 (\lambda_1^2 + \lambda_3^2) \Psi + P;$$
$$\tau^{33} = \lambda_3^3 \Phi + \lambda_3^2 (\lambda_1^2 + \lambda_2^2) \Psi + P;$$
$$\tau^{12} = \tau^{21} = \tau^{31} = 0.\quad (1.2)$$

Here

$$\Phi = \frac{2}{V I_3} \frac{\partial W}{\partial I_1}; \quad \Psi = \frac{2}{V I_3} \frac{\partial W}{\partial I_2}; \quad P = 2 \sqrt{I_3} \frac{\partial W}{\partial I_3},$$

where $W = W(I_1, I_2, I_3)$ is the function of the specific deformation energy, relative to the unit volume of the nondeformed state $B_0$; $I_1 = \lambda_1^4 + \lambda_2^4 + \lambda_3^4; I_2 = \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2; I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2$ are the invariants of the deformation.

If the material is incompressible ($I_3 = 1$), then $W$ is a function only of $I_1, I_2$, while $P$ is a new unknown function. In the absence of the stresses on the horizontal surfaces $z = \pm h_0/2$, in the relations (1.2) one has to take

$$\tau^{33} = 0,\quad (1.3)$$

which is the additional relation, for example, for the determination of $P$.

Because of the absence of body forces and accelerations, the stresses (1.2) satisfy identically the equilibrium equations

$$\tau^{ii} = 0.$$  

Here the double bar denotes covariant differentiation with respect to the state $B$.

2. The deformed state $B'$ is obtained from the state $B$ by the superposition of additional small deformations. The curvilinear coordinates $\theta_i$ are so chosen that they coincide with the rectangular system of coordinates $x, y, z$ in the deformed state $B$

$$\theta_1 = x; \quad \theta_2 = y; \quad \theta_3 = z.\quad (2.1)$$
The general three-dimensional theory of small elastic deformations, superimposed on finite elastic deformations, is described by the following system of equations [5]:

1) the equations of motion linearized with respect to the additional displacements

\[
\frac{\partial^2 \mathbf{u}}{\partial t^2} + \mathbf{u}' \frac{\partial \mathbf{u}}{\partial x} + \mathbf{u}' \frac{\partial \mathbf{u}}{\partial y} + \frac{1}{\rho} \mathbf{w} = \mathbf{q},
\]

where \( \rho = \rho_0 \frac{1}{\varepsilon^2} \); \( \rho_0 \) is the density of the material in the state \( \mathbf{B}_0 \);

\[
w_1 = u_x; \quad w_2 = u_y; \quad w_3 = u_z \tag{2.3}
\]

2) the elasticity relations

\[
\tau_{11} = C_{11} \varepsilon_x + C_{14} \varepsilon_y + C_{15} \varepsilon_z; \\
\tau_{22} = C_{21} \varepsilon_x + C_{22} \varepsilon_y + C_{25} \varepsilon_z; \\
\tau_{33} = C_{31} \varepsilon_x + C_{32} \varepsilon_y + C_{33} \varepsilon_z; \\
\tau_{23} = C_{13} \varepsilon_x; \quad \tau_{13} = C_{23} \varepsilon_y; \quad \tau_{12} = C_{32} \varepsilon_x. 
\]

Here we have introduced the notation:

\[
et_x = \frac{\partial u_x}{\partial x}; \quad et_y = \frac{\partial u_y}{\partial y}; \quad et_z = \frac{\partial u_z}{\partial z}; \\
et_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}; \quad et_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}; \quad et_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}; \\
C_{11} = 2\lambda_1 [A_{11} + (\lambda_2^2 + \lambda_3^2) A_{12} + \lambda_2 \lambda_3 A_{13} + 2(\lambda_2^2 + \lambda_3^2) A_{12}] \\
+ 2\lambda_2 \lambda_3 \lambda_4 A_{13} + 2\lambda_2 \lambda_3 \lambda_5 A_{15} - \tau_{11}; \\
C_{12} = 2\lambda_2 \lambda_3 \lambda_4 [A_{11} + (\lambda_2^2 + \lambda_3^2) A_{12} + \lambda_2 \lambda_3 A_{13} + (\lambda_1^2 + \lambda_3^2) A_{15}] \\
+ 2\lambda_3 \lambda_4 A_{13} + \lambda_1^2 \lambda_2 \lambda_3 A_{15} + \lambda_2 \lambda_3 A_{15} + 2(\lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2) A_{23} \\
- \lambda_1 \lambda_2 \lambda_3 \delta + \lambda_1^3 \lambda_2^2 A_{15} + P; \\
C_{13} = C_{14}; \quad C_{23} = C_{24}; \quad C_{33} = C_{34}; \\
C_{15} = -\lambda_2 \lambda_3 \lambda_4 \delta - P; \quad C_{15}' = -\lambda_1 \lambda_2 \lambda_3 \delta - P; \\
C_{26} = -\lambda_1 \lambda_2 \lambda_3 \delta - P; \quad A_{16} = \frac{2}{V_3} \frac{\partial \mathbf{w}}{\partial \mathbf{h}}.
\]

where the quantities \( C_{11}' \), \( C_{23}' \), and \( C_{31}' \) are obtained from \( C_{11} \), and \( C_{23} \) and \( C_{31} \) from \( C_{12} \), by the cyclic permutation of \( \lambda_1, \lambda_2, \lambda_3 \) and \( \tau_{11}, \tau_{22}, \tau_{33} \).

To the equations presented above one must add the boundary conditions [5].

3. We note that there are many common equations between (2.2)-(2.5) and the equations of the theory of anisotropic plates. Therefore, for the construction of a more precise theory of plates subject to finite initial deformations we consider the following prerequisites [1]:

a) the displacement \( u_z \), normal to the medium plane of the plate, does not depend on the coordinate \( z \);

b) the tangential stresses \( \tau_{13} \) and \( \tau_{23} \) or the corresponding strains \( \varepsilon_{xz} \) and \( \varepsilon_{yz} \) have a prescribed variation along the thickness of the plate;

c) the stress \( \tau_{33} \), normal to areas which are parallel to the median plane, can be neglected in comparison with other stresses.

In the following, in the relations (2.2), (2.4), (2.5) we switch to the physical components of the stress tensor

\[
\sigma_x = \tau_{11}; \quad \sigma_y = \tau_{22}; \quad \sigma_z = \tau_{33}; \\
\tau_{xy} = \tau_{12}; \quad \tau_{xz} = \tau_{13}; \quad \tau_{yz} = \tau_{23}. 
\]

(3.1)