The analysis of the natural vibrations of piezoelectric transducers of comparable dimensions is a very complicated problem of electroelasticity. Individual results obtained by different methods for piezoelectric cylinders with axial and radial polarization can be found in [1, 2, 4, 7, 9]. The finite elements method (FEM) has great potential for analyzing piezoelectric transducers of arbitrary geometry and dimensions, making it possible to consider any type of polarization, electrodes of arbitrary form, and different structural features.

The goal of the present study is to completely analyze the natural vibrations of piezoelectric transducers of the axisymmetric type with arbitrary dimensions. We want to analyze the spectra of natural frequencies in resonance and anti-resonance regimes, the dynamic electromechanical coupling factor (DECF), and the mode of vibration using the example of a radially polarized cylinder.

The system of equations describing the vibrations of a piezoelectric finite element that was obtained in [6] on the basis of the variational principle is best written in dimensionless form with allowance for the following boundary conditions: absence of mechanical stresses at the boundary; absence of the normal component of electrical induction on the surfaces not covered by electrodes. By introducing the appropriate generalized matrices to account for the effect of the electric field, we can write the system of equations in the form of a single matrix equation which coincides in structure with the corresponding equation for an isotropic solid

\[
(k - (k_0)^2 c_{44} |m|) |\omega| = |P_i|.
\]

where \(k_e\) is the wave number of a transverse wave; \(a\) is the geometric dimension for which normalization is performed - in the present case, the mean radius of the cylinder; \(c_{44}\) is an element of the dimensionless matrix of elastic constants of the ceramic which comprises the cylinder; \(|m|\) is the mass element matrix, augmented by zero columns and rows for the variables connected with the electric field; \(k\) is the generalized element stiffness matrix; \(|w_i| = <w_i>/a\) is the dimensionless nodal displacements (here and below, the prime denotes dimensional quantities); \(V_i = V_i' e_3'/e_3\) are dimensionless nodal electric potentials; \(\varepsilon_3'\) is an element of the matrix of piezoelectric constants; \(\varepsilon_3'\) is an element of the matrix of piezoelectric constants; \(f_i = f_i'/(c_3' a)\) are dimensionless nodal forces; \(c_3'\) is an element of the matrix of elastic constants; \(q_i = q_i'(c_3' a)\) are the dimensionless nodal charges; \(\beta = (e_3')^2/(\varepsilon_3' c_3')\);
formed from element matrix (1) by the standard FEM [3] and has the same form if we replace the element matrices by global matrices. After we form the global system of equations, we make allowance for the electrical boundary conditions for the electrodes (which are equipotential surfaces): grounding of the nodes on a given electrode; assignment of the potential for the nodes located on the other electrode. The electric charge is equal to the sum of the charges of all nodes of an electrode and with a harmonic time dependence is expressed through the current in the circuit of the transducer \( q = iI/\omega \), where \( i \) is the imaginary unit and \( \omega \) is the angular frequency. The electrical variables in the global system of equations will include the following: the applied voltage \( V \); the current in the transducer circuit \( I \); the potential of interior nodes not connected with the electrode \( V_i \). If we are interested only in the distribution of the displacements and do not care about the distribution of the electric field, we can significantly reduce the order of the global system of equations by excluding the potentials of the interior nodes (arranging for compensation [3, 8]). Then the global system will have only two electrical variables \( V \) and \( I \). We will write out the main expressions, assuming for the sake of brevity that the global mass matrix \([M]\) and the corresponding coefficients are included in the global stiffness matrix \([K_{uu}]\). We isolate the homogeneous part of the system from the nonhomogeneous part in the absence of external nodal forces

\[
\begin{align*}
|K_{uu}| |K_{uv}| |K_{vv}| u_i & = 0 \\
\langle K_{uv} \rangle & \langle K_{vv} \rangle V = i/\omega, \\
|K_{uv}^\prime | |K_{vv}^\prime | V_i & = 0
\end{align*}
\]

(2)

where \([K_{uv}]\) is the global matrix of piezoelectric "stiffness;" \([K_{vv}]\) is the global matrix of dielectric "stiffness;" the primes denote blocks of global matrices. After performing the appropriate transformations, we have a system of the following form:

\[
\begin{align*}
(|H_{uu} | - (k_a)^2 c_{44} |M|) u_i + |H_{uv} | V & = 0, \\
\langle H_{uv} | u_i \rangle + H_{uv} V & = i/\omega,
\end{align*}
\]

(3)

where

\[
\begin{align*}
|H_{uu}| & = |K_{uu}| - |K_{uv}^\prime | K_{vv}^{-1} |K_{uv}^\prime |^\top, \\
\langle H_{uv} \rangle & = \langle K_{uv} \rangle - \langle K_{uv}^\prime \rangle K_{vv}^{-1} |K_{uv}^\prime |^\top,
\end{align*}
\]

Assuming that the potential of an electrode is known in the resonance regime, we can use Eq. (3) to obtain the following expression for electric conduction: \( 1/Z = 1/V = -i\omega \) \( (H_{vv} - <H_{uv}|A|^{-1}|H_{uv}> \) ), where \( |A| = |H_{uu}| - (k_a)^2 c_{44} |M| \). Conduction approaches infinity (without allowance for loss) if \( \text{det} |A| \to 0 \), i.e., the frequency approaches the resonance frequency. Thus, the resonance frequency can be found by solving a generalized matrix eigenvalue problem

\[
|A| u_i = 0.
\]

(4)