Volterra's idea [40] of applying nonlinear functionals to the construction of the equations of state of materials with memory has recently been intensively developed as a result of the achievements of modern functional analysis. The fundamentals of the modern method of constructing and analyzing the equations of state using time-dependent nonlinear functionals are described in [3].

In this paper we discuss the most recent work in this area (up to papers published in 1968).

In the energy method of deriving the initial physical relations between the stress and deformation components the fundamental equation is

\[ \tau^{ij}(t) = \frac{1}{V I_s} \frac{\partial A}{\partial \gamma_{ij}(t)} , \]

where the \( \tau^{ij}(t) \) are the contravariant components of the stress tensor for the deformed body; the \( \gamma_{ij} \) are the components of the deformation tensor; \( I_s \) is the third scalar invariant of the deformation; \( I_s = |\delta_t^{ij} + 2\gamma_t^{ij}| \); \( \delta_t^{ij} \) is the Kronecker delta; \( \gamma_t^{ij} \) is a mixed component of the deformation.

The functional \( A \left[ \gamma_{ij}(t) \right] \) is the accumulated energy of deformation, depending not only on the deformed state at the given moment but also on the complete history of the deformation. It is assumed that the nonlinear functional has a continuous Frechet derivative of arbitrarily high order. The latter makes it possible to represent the accumulated energy in the form of a series of multiple integrals of Volterra-Stieltjes type.

In [23] the investigation was subject to a restriction in the first and second order of the derivatives. When temperature phenomena and the aging effect are absent the above functional is approximated by linear and quadratic functionals. By enlisting two theorems on polynomial bases we can study the symmetry of the material and establish the appropriate constraints to be imposed on the functional.

From (1) we can obtain equations of state linking the stress and deformation components with the aid of the multiple integrals. In particular we can obtain equations containing two Volterra-type inheritance kernels which are frequently used to solve actual problems.

In the studies of many authors the stress components are determined from the functional directly in terms of the deformation gradients. The object of these studies is to establish the properties of the functionals and their approximations. The approximation of functionals by polynomials in the theory of materials with memory is studied in [26] where, using functional analysis, a theorem is proved establishing the possibility of such an approximation.

The case of linearization in nonlinear viscoelasticity, when the functional linking the stress and the deformation can be represented as a multiple integral series in which terms through the fourth order are retained is described in [8]. Small time-dependent deformations which are added to the finite instantaneous elastic deformations are discussed.
In [7, 38] truncated Volterra multiple integral series are used to solve applied problems. The first paper discusses shear waves of finite deformations on the surface of a viscoelastic half-space, the second, the equivalent linear rheology of rectangular plates.

Papers [36, 37, 32] are devoted to the symmetry of a material which can be described using some group \( \{Q\} \) of time-independent transformations, transforming the given coordinate system into another without changing the properties of the material. The conditions are indicated under which the scalar functional is invariant with respect to the corresponding group \( \{Q\} \). Various particular forms of the functional and the question of its approximate representation as a differential polynomial or in source-like form as a series of multiple integrals corresponding to the cases of symmetry of the material, and, finally, isotropy, are all discussed.

A further extension of the concept of the symmetry of the material was given in [45]. In this connection it is expedient to recall the investigations of [20, 22, 35] and also [39], in which Rivlin's results are applied to study the propagation of acceleration fronts. Papers [9, 12, 34] are in the same class of investigations.

In [12] was established the form of the functional defining the relation between the stress components and the gradients of the deformation components, which satisfies the principle of invariance with respect to rotation of the physical hereditary system. If the stress tensor is symmetric, the functional \( F_k = F_{k'} \). On the basis of this relation were obtained two forms of the equations suitable for describing the deformation processes, taking account of inheritance when small deformations are added to established large deformations. The nature of the constraints to be imposed on \( F_k \) is clarified; if these constraints hold the derived equations describe the deformation process for an elastic-hereditary material which is symmetric in the undeformed state.

In [34] is studied the relation between the independent stress-relaxation moduli and the order of approximation of the hereditary functional corresponding to the number of terms retained in the multiple integral series. It is established that for initially isotropic materials only six independent kernels appear in the third order approximation. If the material is also incompressible, the number of kernels reduces to four.

Various cases of approximating hereditary functionals were discussed in [9] depending on whether the mechanical properties of the material had a particular symmetry. This can occur after mapping the time interval from \(-\infty\) to \( t \) into the interval \((0, 1)\) and applying Weierstrass's theorem on the uniform approximation of these functionals by polynomials in linear continuous functionals depending on the components of the metric tensor.

A number of investigations are devoted to isotropic materials in which the fundamental mathematical method is to expand the functional in a Volterra multiple integral series. Thus, a relation between the stress and deformation tensors was constructed in [31] on the basis of the above expansion, but retaining only the first two terms containing single and double integrals. The equation contains seven inheritance kernels (two of them correspond to linear inheritance) and the three usual invariants of the deformation tensor. An expansion of the functional, defining a five-dimensional stress vector, in terms of a small parameter is given in [2].

The conditions for the asymptotic stability of the solution of the nonlinear equation in inheritance theory containing a Volterra multiple integral series were obtained in [18]. The method of solving the equation with a truncated multiple integral series was developed by Volterra [40] and used in [18]. This method was further developed in [1, 4, 5, 6, 17]. In particular, a fundamental result of [17] is the construction of an algorithm for finding all the resolvent kernels from given relaxation kernels in the multiple integral polynomial defining the stress.

In connection with the study of waves in homogeneous and isotropic materials with memory, detailed investigations were made in [14–17, 29] of the differential properties of four functionals defining in the one-dimensional case the stress, the entropy, the free energy, and the heat flux as functions of the hereditary properties of the deformation gradient and the absolute temperature. These functionals are subject to the principle of fading memory, assuming the existence of continuous Frechet derivatives. The latter makes it possible to define the generalized modulus as the derivative of the initial functional when the stressed state is homogeneous.