APPLICATION OF THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE TO SOLUTION OF A TEMPERATURE PROBLEM IN BENDING OF THIN PLATES

R. N. Shvets


UDC 517.53:539.377

Utilization of functions of a complex variable [5] often permits effective solution of force problems in two-dimensional elasticity theory [7], and of problems in bending of thin elastic plates [3, 7]. Thus this is a desirable approach in solving the corresponding problems of thermoelasticity.

For the two-dimensional problem of thermoelasticity, the method of functions of a complex variable was first considered by N. N. Lebedev [21]. Here the method is extended to temperature problems in bending of thin plates.

§1. We consider a thin isotropic plate of thickness \( h \), located in a temperature field

\[
t = t(x, y, \tau) = -t(x, y, -\tau),
\]

that is antisymmetric with respect to the middle surface; the field causes the plate to bend.

If we use the assumptions ordinarily made in the theory of thin-plate bending [3, 7], to determine the stresses produced in the plate by a deflection \( w \) we find

\[
\begin{align*}
\sigma_{11} &= -\frac{E\gamma}{1-\nu^2} \left( \frac{\partial w}{\partial x} + \nu \frac{\partial w}{\partial y} \right) - \frac{\alpha E}{1-\nu} t, \\
\sigma_{22} &= -\frac{E\gamma}{1-\nu^2} \left( \frac{\partial w}{\partial y} + \nu \frac{\partial w}{\partial x} \right) - \frac{\alpha E}{1-\nu} t, \\
\sigma_{12} &= -\frac{E\gamma}{1+\nu} \frac{\partial w}{\partial x}.
\end{align*}
\]

Accordingly, the bending and torsional moments will be

\[
\begin{align*}
M_1 &= -D \left[ \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} + \frac{\alpha(1+\nu)}{h} T_z \right], \\
H_{12} &= -D (1-\nu) \frac{\partial w}{\partial x}, \\
M_2 &= -D \left[ \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} + \frac{\alpha(1+\nu)}{h} T_z \right],
\end{align*}
\]

where

\[
D = \frac{E h^3}{12 (1-\nu^2)}, \quad T_z(x,y) = \frac{h^2}{k^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} t \, dy,
\]

and \( \alpha \) is the coefficient of linear expansions.

The quantity \( T_z \) occurring in (1.3) is found from the equations [1, 4, 6]

\[
h^3 \Delta T_z - 3(1+\nu) T_z + \frac{h^3}{\alpha} \frac{\partial^2 T_z}{\partial \tau^2} = -3 \mu \psi_{\tau\tau},
\]

the boundary condition at the edge \( L \) of the plate

\[
\frac{\partial T_z}{\partial \tau} + k (T_z - T_c) = 0,
\]

and the initial condition

\[
T_z(x,y,0) = T_s(x,y).
\]

Here \( \mu = 2h\theta_x, \quad T_c = 1/2(t_{c+} - t_{c-}) \), \( t_{c+} \) and \( t_{c-} \) are the temperatures of the external medium washing the plate surface \( y = \pm h/2 \), \( k, \theta \) are the relative heat-transfer coefficients, \( n \) is the external normal to the contour \( L \) of the plate, and \( a \) is the thermal diffusivity.

We eliminate the shear forces \( N_1 \) and \( N_2 \) from the equilibrium equations for a plate element,

\[
\begin{align*}
\frac{\partial M_1}{\partial x} + \frac{\partial H_{12}}{\partial y} - N_1 &= 0, \\
\frac{\partial H_{12}}{\partial x} + \frac{\partial M_2}{\partial y} - N_2 &= 0,
\end{align*}
\]

and, taking (1.3) into account we obtain the inhomogeneous biharmonic equation

\[
\Delta \Delta w = -\frac{\alpha(1+\nu)}{h} \Delta T_z \left( \Lambda = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)
\]

for determining the deflection \( w \). The general solution of (1.8) can be written as

\[
w = \text{Re} \left[ \bar{\psi}(z) + \chi(z) \right] - \frac{\alpha(1+\nu)}{4h} \int T \, d\tau d\zeta,
\]

where \( \bar{\psi}(z) \) and \( \chi(z) \) are analytic functions in complex variable \( z = x + iy \); the second term on the right side of (1.9) is a particular solution of the inhomogeneous equation (1.8).

Substituting (1.9) into (1.3), as well as into the first two equations of (1.7) and remembering that \( \chi'(z) = \bar{\psi}(z) \), after certain manipulations we obtain

\[
M_1 = -D \text{Re} \left[ \frac{(1+\nu)}{h} \psi'(z) + \frac{1}{2} \text{Im} \left( \bar{\psi}(z) + \psi(z) \right) \right] + ...
\]

*Study reported to the fifth all-union conference on plate and shell theory, held 3–6 February 1965 in Moscow.
\[ + o (1 - v) \left[ \int \frac{\partial T_2}{\partial z} dz + \int \frac{\partial T_3}{\partial z} dz - 2 T_2 \right], \]

\[ M_n = - D Re \left( (1 + v) \psi (z) - \frac{1}{2} \left[ \phi \psi (z) + \psi \phi (z) \right] \right) - \]

\[ - D \frac{\alpha (1 - v)}{4h} \left[ \int \frac{\partial T_2}{\partial z} dz + \int \frac{\partial T_3}{\partial z} dz + 2 T_2 \right], \]

\[ H_{13} = - iD \frac{1 - v}{2} \text{Im} [\phi \psi (z) + \psi \phi (z)] + \]

\[ + \psi (z) + iD \frac{\alpha (1 - v)}{4h} \left( \int \frac{\partial T_2}{\partial z} dz - \int \frac{\partial T_3}{\partial z} dz \right), \]

\[ N_1 = - 2D [\psi (z) + \phi (z)], \]

\[ N_2 = - 2D [\phi (z) - \psi (z)] \]  

(1.10)

or

\[ M_1 + M_2 = \]

\[ = - 2 (1 + v) D \left[ \psi (z) + \phi (z) + \frac{\alpha (1 - v)}{2h} T_2 \right], \]

\[ M_3 - M_1 + 2iH_{12} = \]

\[ = 2 (1 - v) D \left[ - 2 \psi (z) + \phi (z) - \frac{\alpha (1 + v)}{2h} \int \frac{\partial T_2}{\partial z} dz \right], \]

\[ N_1 - iN_2 = - 4D \phi (z). \]  

(1.11)

The functions \( \phi (z) \) and \( \psi (z) \) are found from the conditions at the plate contour. If the moments and forces are specified as a function of the arc \( s \) at the contour \( L \),

\[ M = m(s), \quad N = \frac{\partial H}{\partial s} = p(s), \]  

(1.12)

we then have

\[ m \phi (z) + z \phi (z) + \psi (z) = \]

\[ = \frac{\alpha (1 + v)}{2h} \int T_2 dz + f_1 + if_2 + izC + C^1 \]  

(1.13)

to determine \( \phi (z) \) and \( \psi (z) \); here \( m_0 = -(3 + v)/(1 - v) \), while \( C \) and \( C^1 \) are real and complex variables;

\[ f_1 + if_2 = \frac{1}{D(1 - v)} \int \left[ m(s) + i \int p(s) ds \right] ds. \]  

(1.14)

If the deflections \( w \) and rotation angles \( dw/dn \) are specified on the plate contour, the corresponding boundary condition on \( L \) has the form

\[ \phi (z) + z \phi (z) + \psi (z) = \]

\[ = \frac{\alpha (1 + v)}{2h} \int T_2 dz + f_1 + if_2. \]  

(1.15)

Here \( f_3 + if_4 = e^{i\theta_3} (dw/dn + d\omega/ds) \), \( \theta_3 \) is the angle between the normal \( n \) and the ox-axis, and \( w(s) \) is a known function.

For the mixed problem, \( \phi (z) \) and \( \psi (z) \) must satisfy (1.13) on that portion of the contour where the moment and forces are specified, while (1.15) must be satisfied on the portion of the contour where the values of \( w(s) \) and \( dw/dn \) are specified.

\section{2. We now consider the problem of stationary temperature fields that do not produce thermal stresses in a free plate. The stresses in a bent plate are found from the formulas}

\[ \sigma_{11} = \frac{12Y}{h^2} M_1 + \frac{\alpha E}{1 - \nu} \left( \frac{\nu}{h} T_2 - t \right), \]

\[ \sigma_{22} = \frac{12Y}{h^2} M_2 + \frac{\alpha E}{1 - \nu} \left( \frac{\nu}{h} T_2 - t \right), \]

\[ \sigma_{12} = \frac{12Y}{h^2} H_{12}. \]  

(2.1)

Letting \( \sigma_{11} = \sigma_{12} = 0 \) in (2.1), we obtain

\[ \frac{12Y}{h^2} M_1 + \frac{\alpha E}{1 - \nu} \left( \frac{\nu}{h} T_2 - t \right) = 0, \]

\[ \frac{12Y}{h^2} M_2 + \frac{\alpha E}{1 - \nu} \left( \frac{\nu}{h} T_2 - t \right) = 0, \]

\[ H_{12} = 0. \]  

(2.2)

Equations (2.2) are possible only if

\[ t = \frac{\nu}{h} T_2, \quad M_1 = M_2 = H_{12} = 0. \]  

(2.3)

As a consequence, the temperature stresses in the plate will be zero if and only if the temperature is a linear function of the coordinate \( \gamma \) and the temperature moments are zero.

By virtue of (1.11) and the equilibrium equations (1.7), the requirement that the moments equal zero yields

\[ \tilde{\phi} \psi (z) + \tilde{\psi} \phi (z) = \frac{\alpha (1 + v)}{2h} \int \frac{\partial T_2}{\partial z} dz, \]

\[ \tilde{\phi} \psi (z) + \tilde{\psi} \phi (z) = - \frac{\alpha (1 - v)}{2h} T_2, \quad \phi (z) = 0, \]  

(2.4)

and from this it follows that

\[ T_2 = \text{const}, \]

\[ \psi (z) = - \frac{\alpha (1 - v)}{2h} Az + C^2, \]

\[ \psi (z) = C_2, \]  

(2.5)

where \( A = T_1 + iC_2, \) \( C_2 \) is a real constant, and \( C_1 \) and \( C_2 \) are complex constants.

Substituting (2.5) into the equation

\[ u + iv = \frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} = \]

\[ = \phi (z) + \frac{\alpha (1 + v)}{h} \int T_2 dz, \]  

(2.6)