CONFIGURATIONS WITH A VECTOR FIELD IN THE EINSTEIN—CARTAN THEORY

V. G. Krechet

The properties of self-gravitating distributions of an ideal fluid charged by a vector field which is either massive or massless, i.e., electromagnetic, are considered taking account of the spin properties of the vector field in the framework of the Einstein—Cartan theory of gravitation. Conditions for equilibrium are found and the corresponding exact solutions are obtained. A complete system of first integrals of the corresponding equations of motion is found for dynamical distributions in the absence of pressure. A theorem on the correspondence between the dynamics of an electrically charged ideal fluid with a limiting equation of state and the dynamics of a free massive vector field is also proved.

At the present time the effect of a distributed electric charge on the dynamics of self-gravitating distributions of an ideal fluid in the case of zero pressure has been studied rather completely in the GTR [1–4, et. al.]. Certain important aspects of the behavior of self-gravitating distributions taking into account the pressure and also distributions charged by a massive vector field have also been studied [5–7]. However, within the framework of GTR it is not possible to account for the effect of a proper angular momentum (spin) of vector fields which are either massive or massless (the electromagnetic field). In the present paper, in the framework of the Einstein—Cartan theory, certain aspects of the dynamics (statics) of spherical configurations of an ideal fluid charged by a vector field are considered with account of the effect of the spin of this field.

1. The Basic Equations

In the Einstein—Cartan theory a model of the space—time manifold is a four-dimensional space with an affine connection and a pseudo-Riemannian metric $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ and torsion together with the metric condition (the space $U_a$)

$$\nabla_a g_{\alpha\beta} = 0.$$  

(1)

Under condition (1) the coefficients of the connection are found from the expression

$$\Gamma^\gamma_{\alpha\beta} = \{\gamma^\gamma_{\alpha\beta}\} + Q^\gamma_{\alpha\beta} + Q^\gamma_{\alpha\beta} + Q^\gamma_{\alpha\beta},$$  

(2)

where $Q^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}$ are the components of the torsion tensor. The stress tensor $F_{\alpha\beta}$ of a vector field is defined by

$$F_{\alpha\beta} = 2\nabla_\alpha A_\beta = \partial_\alpha A_\beta - \partial_\beta A_\alpha - 2Q^\gamma_{\alpha\beta} A_\gamma.$$  

(3)

In general form the compatible system of equations of the gravitational field, an ideal fluid, and a vector field are:

a) $R_{\alpha\beta} - \frac{1}{2} Rg_{\alpha\beta} = \kappa [(\rho + p) U_\alpha U_\beta - pg_{\alpha\beta}] + 2\kappa [F_{\alpha\gamma} F^{\alpha\gamma} g_{\beta\gamma}/4 - F_{\alpha\beta} F^{\alpha\beta} + (\nabla_\alpha + 2Q^\gamma_{\alpha\beta}) A_\gamma F^{\gamma\beta} + \mu^2 A_\alpha A_\beta - \mu^2 A_\alpha A_\beta g_{\gamma\beta}/2];$

$$+ (\nabla_\alpha + 2Q^\gamma_{\alpha\beta}) A_\gamma F^{\gamma\beta} + \mu^2 A_\alpha A_\beta - \mu^2 A_\alpha A_\beta g_{\gamma\beta}/2];$$

b) $(\nabla_\alpha + 2Q^\gamma_{\alpha\beta}) F^{\alpha\beta} - \mu^2 A^\lambda = - 4\pi \rho U^\lambda;$$

c) $Q^\gamma_{\alpha\beta} + 2\gamma^\gamma_{\alpha\beta} Q_{\gamma\beta} = 2\kappa A_\gamma F^{\gamma\beta}.$

Here $\varepsilon$, $p$, $\rho$ are respectively the energy density of the fluid, the pressure, and the charge density; $U$ is the $4$-velocity of the fluid; $(1/4\pi)A_0 [\nu F_\nu]_\xi$ is the tensor of the spin density of the vector field; $\beta = \pi/8\pi = G/c^4$.

2. Static Configurations of a Charged Ideal Fluid

By eliminating the torsion by means of Eq. (4c) for the case of equilibrium, in the presence of spherical symmetry the system (4) can be brought to the form

$$e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) = -8\pi\varepsilon - 2\beta^2 e^{-\lambda} - \beta^2 \nu^2 e^{-\lambda} + \frac{\beta^2 \nu^2 e^{-\lambda}}{(1 + \beta^2 \nu^2 e^{-\lambda})^2} \phi^2 e^{-\lambda}, \tag{5a}$$

$$e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) = \beta^2 \nu^2 e^{-\lambda} + \beta^2 \nu^2 e^{-\lambda} + \frac{\beta^2 \nu^2 e^{-\lambda}}{1 + \beta^2 \nu^2 e^{-\lambda}} \phi^2 e^{-\lambda}, \tag{5b}$$

$$e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{\nu'}{r} = \beta^2 \nu^2 e^{-\lambda} + \beta^2 \nu^2 e^{-\lambda} + \frac{\beta^2 \nu^2 e^{-\lambda}}{1 + \beta^2 \nu^2 e^{-\lambda}} \phi^2 e^{-\lambda}, \tag{5c}$$

$$\left[ \frac{\nu'}{r} + \frac{\nu^2}{2} - \frac{\nu^2}{4} \phi^2 e^{-\lambda} + \frac{\nu^2}{2r} \right] = \beta^2 \nu^2 e^{-\lambda} + \beta^2 \nu^2 e^{-\lambda} + \frac{\beta^2 \nu^2 e^{-\lambda}}{1 + \beta^2 \nu^2 e^{-\lambda}} \phi^2 e^{-\lambda}. \tag{5d}$$

Here $\lambda$, $\nu$ are metric coefficients; $\phi$ is the potential of the vector field, while terms of second order of smallness and in higher in the small parameter $\beta = \pi/8\pi$ in the equations of the gravitational field (5a)-(5c) and of first order of smallness in the equation of the vector field (5d) describe the contribution of torsion.

We first investigate the possibility of equilibrium in the absence of pressure ($p = 0$) when the ratio of the charge density to the energy density is the same at all points of the configuration in question:

$$\rho/\beta \cdot \varepsilon = \text{const} \equiv a. \tag{6}$$

Equations (5) together with (6) form a closed system for determining the five unknown functions $\lambda$, $\nu$, $\varepsilon$, $\rho$, $\phi$, which implies the relation

$$\frac{a^4}{(1 + a^2)^3} \frac{\nu^2}{4} e^{-\lambda} + 4\pi\sqrt{a} \cdot \varepsilon = \frac{\mu^2(1 - a^2)}{a^2}. \tag{7}$$

Since the left side of this equation is strictly positive, two important conclusions can be drawn from (7):

a) if $\mu^2 = 0$, i.e., for the case of an electrically charged dust, under the condition (6) there does not exist an equilibrium state in contrast to the corresponding situation in GTR where equilibrium exists for $a = 1$;

b) if $\mu^2 \neq 0$, then there is equilibrium under the condition $a < 1$. For comparison we point out that in the absence of torsion, equilibrium is possible under the condition $1 < a < \sqrt{2}$ [7]. Further, for $\mu^2 \neq 0$, proceeding as in [7], we go over to the new variables

$$r = e^{2\lambda}; \quad y = \frac{A(x)F(x)}{\sqrt{A^2 - a^2(1 + a^2)}}, \quad F^2(x) = \frac{a^2}{1 + a^2} \left[ e^{2\lambda} + y d^2 e^{2\lambda} \right]; \quad z^2 \equiv \left[ \frac{1}{a^2} \right]. \tag{8}$$

The systems (5) and (6) can then be reduced to a single Abel equation for the function $\lambda(x)$:

$$A' = 1 - \frac{F'}{F} A - \frac{1 + a^2}{a^2} A^2 + \frac{(1 + a^2)}{a^2} \frac{F'}{F} A \left( A' = \frac{dA}{dx} \right), \tag{9}$$

which up to numerical coefficients coincides with the corresponding equation obtained within the framework of GTR in [7] and belongs to a type not integrable by quadratures [8].

Returning again to the system (5), it is easy to show that there exists a situation where the effect of the spin on the gravitational field fully compensates for the contribution of the energy tensor of the ideal fluid, so that for $\mu = 0$ the gravitational field inside the