SEPARATION OF VARIABLES IN THE
KLEIN — GORDON EQUATION. I

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All types of external electromagnetic fields containing arbitrary functions which admit of
separation of variables in the Klein—Gordon equation by using three first-order differential
symmetry operators, and stationary fields admitting separation of variables by using two
first- and one second-order differential operators, are found. The curvilinear coordinates
in which the variables are divided are presented and the equations are written down in the
separated variables.

The method of separation of variables is one of the fundamental mathematical methods for finding
the exact solutions of mathematical physics problems. Its role is especially great in the exact solution of
quantum mechanics problems since practically all known exact solutions of quantum-mechanical problems
are based on this method. In this connection, the problem arises of specific classification of those quan-
tum mechanics problems for which the method of separation of variables is generally applicable. This
problem has been solved for an electron moving in external stationary electromagnetic fields (Schrodinger
equation) in nonrelativistic quantum mechanics. A list of results for the scalar potential case can be found
in [1], and a detailed analysis is given in [2]. A solution of the problem in the presence of a vector poten-
tial has been obtained in [3].

The problem of separation of variables in relativistic quantum mechanics problems has been in-
vestigated much less. Here there are only individual examples of exact solutions and any classification
is completely missing. We have examined the Klein—Gordon equation for a charge (spinless particle)
moving in external magnetic fields, and have solved the problem of seeking all kinds of external fields
admitting complete separation of variables. Coordinate systems in which the Klein—Gordon equation is
separated have also been found, and systems of ordinary differential equations in the separated variables
are written down explicitly.

General definitions and theorems are formulated in this first part of the paper and results of specific
computations are presented for the following cases: a) the complete set of motion integrals (symmetry
operators) consists of three first-order differential operators; b) the complete set consists of two firs-
and one second-order operators and the fields are stationary. The remaining cases will be considered
later. Let us note that all the known exact solutions of the Klein—Gordon equation, besides that found in
[5], are suitable for the classification presented herein.

Let us write the Klein—Gordon equation in an arbitrary curvilinear coordinate system as

\[ (g^{\mu\nu} P_\mu P_\nu - m^2) \psi = 0, \quad P_\mu = i\nabla_\mu + A_\mu, \quad \kappa, \ l = 0, 1, 2, 3, \]  

where \( \nabla_\kappa \) is the covariant derivative; \( A_\kappa \) are the covariant components of the electromagnetic potential
(the notation and system of units are taken from [6]).

Let us formulate the definitions and theorems on whose basis specific computations were carried out.*

*Here we limit ourselves just to formulation of theorems. The mathematical aspect of the theory is ana-
lized in detail in [4].

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I. Let be given an equation of nonparabolic type

\[ H \Phi \equiv \left[ g^{\alpha \beta} P_\alpha P_\beta + v(x) \right] \Phi = \epsilon \Phi. \]  

(2)

where \( P_\kappa = \nabla_\kappa + A_\kappa(x) \), where \( \nabla_\kappa \) is the covariant derivative with respect to \( x^\kappa \) in \( n \)-dimensional Riemannian space with the metric tensor \( g_{\mu \nu}\). The tensor \( g_{\mu \nu} \), the vector \( A_\kappa \) and the scalar \( v(x) \) are independent of the parameter \( \varepsilon \). Later in this section we take the convention:

\[ N + N_0 \leq n; \quad 0 \leq N_0 \leq N \leq n - 1; \]
\[ l, j, k, l = 1, 2, \ldots, n; \quad p, q = 1, 2, \ldots, N; \quad a, \mu, \nu = N + 1, \ldots, n; \]
\[ r = N + 1, \ldots, N + N_0; \quad s = N + N_0 + 1, \ldots, n. \]

Summation is over the repeated subscripts (certainly not double!).

Let us examine a system of differential equations of a special kind

\[ \partial_\alpha \Phi = \lambda_\alpha \Phi, \quad H_\alpha \Phi = \Phi_\alpha \lambda_\alpha \Phi, \quad \Phi_\alpha = \partial_\alpha \Phi, \]

(3)

where \( \Phi_\alpha = \psi_\alpha (x), \quad \text{det}(\psi_\alpha) = 0, \quad \lambda_\alpha = \varepsilon, \)

\[ H_\alpha = \partial_\alpha \partial_\beta + 2\partial_\alpha \partial_\beta + \partial_\gamma \partial_\delta \partial_\gamma + h_\alpha. \]

All the coefficients of the operator \( H_\alpha \) are functions just of \( x^\nu \); and \( \lambda_\alpha \) are arbitrary independent parameters.

**Definition 1.** Equation (2) admits complete separation of variables of the type \( (N_0, N) \) in the coordinate system \( (x) \) which is the same for all values of the parameter \( \varepsilon \) if there exists a system of the form (3), each solution of which turns (2) into an identity in the variables \( (x) \) and the parameters \( \lambda \).

For an elliptic equation \( N_0 = 0 \), hence we speak of separation of variables of the kind \( N \).

The possibility of separating variables is determined by the existence of a complete set of symmetry operators in the given equation (from the physical viewpoint, symmetry operators are interpreted as motion integrals).

**Definition 2.** The set of operators

\[ Y = \gamma^i(x) P_i + \gamma(x), \quad X = X^i(x) P_i + a^i(x) Y + \gamma(x) \]

(4)

will be called a set of \((N_0, N)\) type if the conditions:

a) \( [X, Y] = [X, X] = [Y, Y] = 0; \)
\[ \gamma^i, \mu^p, p^q \]

b) the tensors \( X_{ij} \) and \( Y_{ij} \) are linearly independent;

\[ c^\alpha_{\gamma^i} \gamma_{\alpha} \gamma_{p^q} \]
c) functions \( c^\alpha_{\gamma^i} \gamma_{\alpha} \gamma_{p^q} \), \( b_{\gamma^i} \gamma_{p^q} \), \( c^\gamma \) exist such that \( X_{ij} \gamma_j = c^\alpha_{\gamma^i} \gamma_{\alpha} \gamma_{p^q} \), \( X_{ij} \gamma_j = c^\alpha_{\gamma^i} \gamma_{\alpha} \gamma_{p^q} + b_{\gamma^i} \gamma_{p^q} \). The matrix

\[ c^\alpha_{\gamma^i} = c^\alpha_{\gamma^i} \gamma_{\alpha} \gamma_{p^q} \]

is positive-definite and

\[ c^\alpha_{\gamma^i} c^\alpha_{\gamma^j} = c^\alpha_{\gamma^i} c^\alpha_{\gamma^j} ; \]
\[ c_{\gamma^m} c_{\gamma^m} = c_{\gamma^m} c_{\gamma^m} ; \]

\( c^\alpha_{\gamma^i} c^\alpha_{\gamma^j} \)
d) the rank of the matrix \( Y_{ij} \) is \( N - N_0 \) are satisfied. Let us say that the operator \( H \) admits a set

\[ \gamma^i, \mu_{p^q}, \alpha \]
of type \((N_0, N)\) if there are numbers \( a_{\gamma^i}, a_{\gamma^i}, a_{p^q}, a \) such that

\[ H = a, \gamma^i + a_{p^q} Y_{ij} + a_{p^q} Y_{ij} + a. \]

Without limiting the generality, it can be considered that \( H = X. \)

Let us call the equation \( H \Phi = \epsilon \Phi \) the function-equivalent to (2) if function \( f(x) \) independent of \( \varepsilon \) exists such that \( H = f^{-1} H f \). The passage to a function-equivalent equation is evidently equivalent to a gradient transformation of the vector \( A_\kappa \) but does not alter the scalar \( v \) and the tensors \( g_{\mu \nu} \) and \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). The fundamental theorem can now be formulated thus.