THE CORRESPONDENCE PRINCIPLE BETWEEN CLASSICAL AND QUANTUM QUANTITIES

Yu. I. Zaparovannyi

Questions of the reestablishment of a classical quantity from the known quantum operator are investigated in this article, in that case when the operators are constructed according to the rule proposed in article [15]. The existence of a twofold limiting transition is proved, in which the investigated rule satisfies the requirements of the correspondence principle.

In classical mechanics the specification of the coordinates \( q (q_1, q_2, \ldots, q_N) \) and the momenta \( p(p_1, p_2, \ldots, p_N) \) at the instant of time \( t \) completely determines the state of a system with \( N \) degrees of freedom in the sense that the value of any physical quantity \( A \), characterizing the system under consideration, can be calculated as the value of a certain function \( A(q, p, t) \) [1].

In quantum mechanics the state of such a system is completely determined by the specification of the wave function \( \Psi(q, t) \) [2] in the sense that the experimentally measurable value \( \langle A \rangle \) of the quantity \( A \) at the moment of time \( t \) can be calculated from the known \( \Psi \) with the aid of the following formula:

\[
\langle A \rangle = \int \Psi^* (q, t) O(A) \Psi(q, t) \, dq,
\]

where \( O(A) \) is the operator, representing the physical quantity \( A \) in quantum mechanics. Here and in what follows \( dq = dq_1 dq_2 \ldots dq_N \), and the integration over all variables runs from \(-\infty\) to \(+\infty\).

Understanding by the correspondence principle a connection of physical theories by means of a limiting asymptotic transition with respect to a certain characteristic parameter of certain laws into others [3], it is natural, on the basis of this principle, to require the existence of a definite relationship between the operator \( O(A) \) and the classical function \( A(q, p, t) \). In fact, in the limiting transition from quantum mechanics to classical mechanics, the operator \( O(A) \) must change into the classical function \( A(q, p, t) \) since \( O(A) \) and \( A(q, p, t) \) represent the same physical quantity. In this connection the interrelationship of the operators in quantum theory should, of course, in the limiting transition lead to an analogous interrelationship between the corresponding classical functions. In other words, the operator \( O(A) \), representing the configuration of operators \( O(A_i) \), \( i = 1, 2, \ldots, n \), symbolically written in the form

\[
O(A) = f(O(A_1), O(A_2), \ldots, O(A_n)),
\]

must correspond in the limiting transition to the same configuration of functions, i.e., to the function:

\[
A(q, p, t) = f(A_1(q, p, t), A_2(q, p, t), \ldots, A_n(q, p, t)).
\]

Furthermore, the correspondence principle must manifest itself in the interrelationships of quantities, playing a fundamental role in the evolution of the system, the conservation laws, etc. Thus, for example, in investigating the dynamics of the system in classical mechanics we use the classical Poisson brackets:

\[
\{A(q, p, t), B(q, p, t)\} = \sum_i^N \left( \frac{\partial A(q, p, t)}{\partial p_i} \frac{\partial B(q, p, t)}{\partial q_j} - \frac{\partial A(q, p, t)}{\partial q_j} \frac{\partial B(q, p, t)}{\partial p_i} \right).
\]

In quantum mechanics, the quantum Poisson brackets

\[
\{O(A), O(B)\} = \frac{i}{\hbar} (O(A) \cdot O(B) - O(B) \cdot O(A))
\]

play a no less important role in the dynamics of the system. The correspondence principle requires that the analog of the quantum Poisson bracket (5) in classical mechanics should be the classical Poisson bracket (4).

It is quite clear that the fulfillment of the correspondence principle, in the sense indicated above, depends on the law used to construct the operator \( O(\mathcal{A}) \) for a given physical quantity \( \mathcal{A} \). For example, the correspondence principle for the Poisson brackets associated with the rules for the construction of the operators according to Dirac [4] is always satisfied. However, the Dirac rule is not unique [5, 6], which leads to contradictions [5] associated with its generalization to all possible functions \( A(q, p, t) \). Upon the construction of quantum operators with the aid of von Neumann's rule [7], Weyl's rule [8], the symmetrization rule [9], and the other well-known rules [10-14], the indicated principle is satisfied in the limit \( \hbar \to 0 \).

In the present work the conditions of the limiting transition are investigated, ensuring the fulfillment of the correspondence principle in the sense indicated above in that case when the quantum operators are constructed according to the rule proposed in article [15].

The investigated rule is based on the introduction of a certain set of quadratically integrable functions of coordinates and time \( \varphi_k(q, t) \), satisfying the following normalization condition:

\[
\sum_k \int |\varphi_k(q, t)|^2 dq = 1. \tag{6}
\]

The action of the operator \( O(\mathcal{A}) \) on some function \( U(q, t) \) depends on the chosen set \( \varphi_k \) and is determined by the relation:

\[
O(\mathcal{A}) U(q, t) = (2\pi\hbar)^{-N} \int A_0(q, p, t) e^{\frac{i}{\hbar} (q-p) \varphi} U(q', t) dq' dp. \tag{7}
\]

Here and in what follows, \( (q-p) \) denotes the scalar product of the vectors \( q \) and \( p \). The function \( A_0(q, p, t) \), the so-called generating function of the operator \( O(\mathcal{A}) \) [16], is related to the classical function \( \mathcal{A}(q, p, t) \) by the following integral transformation:

\[
A_0(q, p, t) = \int \varphi(q, p, t) \mathcal{A}(q + \xi, p + \eta, t) d\xi d\eta. \tag{8}
\]

where

\[
\varphi(q, p, t) = (2\pi\hbar)^{-\frac{N}{2}} e^{-\frac{i}{\hbar} (qp)} \sum_k \varphi_k(q, t) \varphi_k^*(p, t) \tag{9}
\]

and the following Fourier transform is to be understood by \( \tilde{\varphi}_k(p, t) \):

\[
\tilde{\varphi}_k(p, t) = (2\pi\hbar)^{-\frac{N}{2}} \int \varphi_k(q, t) e^{-\frac{i}{\hbar} (qp)} dq. \tag{10}
\]

The generating function \( A_0(q, p, t) \) is uniquely related to the operator \( O(\mathcal{A}) \). In fact, assuming \( U(q, t) = \exp \left\{ i q p / \hbar \right\} \) in Eq. (7) and multiplying by \( \exp \left\{ -i q p / \hbar \right\} \), after integration we obtain:

\[
A_0(q, p, t) = e^{-\frac{i}{\hbar} (qp)} O(\mathcal{A}) e^{\frac{i}{\hbar} (qp)}. \tag{11}
\]

In contrast to previously known correspondence rules [4, 7-14], the correspondence rule (7) gives, independently of the form of \( \varphi_k(q, t) \), Hermitian operators for real functions \( \mathcal{A}(q, p, t) \), which are single valued for given \( \varphi_k \), and this rule guarantees that the average values \( \langle \mathcal{A} \rangle \) will be nonnegative provided \( A(q, p, t) \geq 0 \) (see [15, 16]).

It is not difficult to show that the investigated correspondence rule admits an inverse transformation in the sense that the classical function \( \mathcal{A}(q, p, t) \) may be found from the known operator \( O(\mathcal{A}) \). In order to prove the given assertion, we introduce into consideration the Fourier transform \( \tilde{\varphi}(u, v, t) \) of the auxiliary function (9), so that

\[
\varphi(q, p, t) = (2\pi)^{-2N} \int \tilde{\varphi}(u, v, t) e^{i(uq + vp)} du dv, \tag{12}
\]

then introducing the following definitions:

\[
\tilde{\varphi}^{-1}(u, v, t) = \frac{1}{\varphi(u, v, t)}, \tag{13}
\]

\[
\varphi^{-1}(q, p, t) = (2\pi)^{-2N} \int \tilde{\varphi}^{-1}(u, v, t) e^{i(uq + vp)} du dv \tag{14}
\]