THEORY OF FLUCTUATIONS IN QUASIHARMONIC OSCILLATORS

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An approach is developed for analyzing fluctuation effects in quasiharmonic oscillators formed by connecting a nonlinear inertialless element to a resonator. The effect of the nonlinear element on the resonator is treated as a pulsed process. It is assumed that the effect of noise on the system destroys the coherence and changes the response parameters of the resonator to pulse effects, this, ultimately, being responsible for the amplitude and frequency fluctuations. An equation is obtained that describes the fluctuations of the oscillation period. A detailed study is made of the fluctuations of the period in an electronic oscillator due to white noise.

The model of a quasiharmonic oscillator describes the main properties of very many diverse systems of both practical and theoretical interest. This is why, in the study of such oscillators, great attention is devoted to the description of fluctuation effects and a study of their influence on the spectrum of the oscillator. The theory of fluctuations in quasiharmonic oscillators is developed basically for application to electronic generators of harmonic oscillations, which have the greatest practical importance.

In this paper we present an approach differing from [1-7] to the study of fluctuation effects in quasiharmonic oscillators; in a number of cases it has advantages over the well-known approaches through its ability to take into account the features of the oscillating system. The method is set forth below for the electronic model of an oscillator shown in Fig. 1, in which NI is a nonlinear inertialless element that feeds the resonator R with the following equivalent parameters: inductance L, capacitance C, and loss resistance r. The dashed line shows the feedback circuit which controls the nonlinear element.

In the general case, the current feeding the resonator is a sequence of pulses of definite form. Since we consider a quasiharmonic oscillator, Q of the resonator at all times can be assumed fairly large and the resonator treated as a linear inertial circuit with constant parameters.

Each current pulse produces in the resonator a voltage response in the form of a damped sinusoid. At any instant of time, the voltage u(t) across the resonator is the sum of the responses to all the current pulses preceding this time. If the nonlinear element is controlled by the voltage u(t), the time at which the successive pulses arise is uniquely determined by the time at which u(t) reaches the maximal (minimal) value.

If there are no fluctuation sources in the circuit, all the current pulses have the same form and parameters and follow one another after equal time intervals T, which determine the oscillation frequency. The voltage responses in the resonator due to such pulses are added coherently. If there are fluctuation sources, the voltage across the resonator contains a fluctuation component u_n(t), which, when added to u(t), distorts the profile and changes the parameters of the current pulses and also changes randomly the positions of the maxima of u(t) and, therefore, the positions of the current pulses on the time axis. In this case, the voltage responses do not have the same parameters and are added incoherently.

Fig. 2. Coherent addition of resonator responses to pulses in the absence of noise.

Thus, the fluctuations of the parameters and the positions on the time axis of each current pulse are due to both the direct influence of \( u(t) \) on the nonlinear element and the fluctuations of the parameters and the positions of the preceding current pulses, which cause the voltage responses in the resonator to differ and be added incoherently.

We give the treatment for the following case, frequently encountered in practice, when the current feeding the resonator is a sequence of peaked cosinusoidal pulses characterized by a cutoff angle \( \phi \) and maximal value \( I_m \) (see Fig. 2). The response voltage for such a pulse can be obtained by means of the Duhamel integral

\[
U_n(t) = \frac{I_m \omega_0 L}{1 - \cos \phi} e^{-\phi (t-t_n)} [A \cos \omega_0 (t-t_n) + B \sin \omega_0 (t-t_n)],
\]

(1)

where \( \omega_0 \) is the proper frequency of the resonator; \( \alpha = \pi/2L \); \( t_n \) is the instant of time corresponding to the pulse peak; and \( A \) and \( B \) for the quasiharmonic oscillator considered here are given by the approximate expressions

\[
A \approx \sin \phi \cos \phi, \quad B \approx (2 \phi \cos 2\phi - \sin 2\phi) / 4 \omega_0,
\]

(2)

which can be obtained under the assumption that \( Q \) of the resonator is fairly large (20-200), and \( \phi \leq \pi \).

An elementary investigation of the function (1) shows that its successive maxima follow one another at time intervals \( T_0 = 2\pi/\omega_0 \). An exception is the shortened first period, equal to \( T_0 - \Delta T_b \), where

\[
tg \omega_0 \Delta T_b = \frac{2 \omega_0 - B A}{1 - (2B) (\omega_0 A)}.
\]

(3)

For practically important cases \( 30^\circ < \phi < 180^\circ \), and in accordance with (1) and (2) \( B/A = (-1 \text{ to } 0.5) \alpha/\omega_0 \), and \( \tan \omega_0 \Delta T_b \approx (0.5-2) \alpha/\omega_0 \). Since \( \alpha/\omega_0 = 1/2Q = 2.5 \cdot 10^{-2} \text{ to } 2.5 \cdot 10^{-2} \) for \( Q = 20-200 \), one can set

\[
tg \omega_0 \Delta T_b \approx \omega_0 \Delta T_b
\]

(4)

If there are no fluctuations in the circuit, the shortening of the first period of each response will mean that the voltage responses will add coherently but with a time shift \( \beta \) relative to one another, as is shown in Fig. 2. It is not difficult to show that \( \beta \) determines a regular correction to the autooscillation frequency:

\[
\beta = T_0 - T
\]

(5)

Assuming for simplicity that the current pulse peak coincides with the corresponding maximum of the voltage \( u(t) \), we determine the position of the peak of the \( (N+1) \)-th current pulse. Over a time interval of order \( T_0/2 \), the voltage responses can be assumed in a first approximation to be cosineoids shifted relative to one another in phase by \( \omega_0 \beta \) and with amplitude

\[
U = U_0 e^{-\alpha \tau_c (N-1)} = \frac{I_m \omega_0 L A_0 e^{-\alpha \tau_c (N-1)} \exp \left(-\frac{1}{2} \omega_0 \Delta T_b t\right)}{1 - \cos \phi} e^{-\phi \tau_c (N-1)},
\]

(6)

where \( A_0 = \sqrt{A^2 + B^2} \). If the time is measured from the peak of the voltage response to the \( N \)-th current pulse, the voltage \( u(t) \) in a time interval \( \sim T_0/2 \), which includes the \( (N+1) \)-th current pulse, can be written in the form