The Green's function for an electron in the field of a quantized free monochromatic wave is found. A method is proposed for constructing a complete orthonormal system of wave functions for this problem.

The problem of an electron in the field of a free, quantized, monochromatic electromagnetic wave, with the electron—wave interaction taken into account exactly, was first taken up by Berson [1]. The solutions found there were subsequently generalized by various investigators to more complicated cases (see, e.g., [2] and the literature cited there). However, even in the simplest system, studied in [1], several specific problems arise: First, a "forbidden" zone arises in the momentum space of an electron in the field of a quantized wave. The values of the momentum in this "forbidden" zone in the equivalent classical problem lead to unstable oscillators — the reason for the "forbiddenness." Second, a study of the position of the "forbidden" zone rapidly leads to the conclusion of a violation of charge symmetry in the Berson functions [2]. Finally, yet another problem is the lack of a proof that the system of these functions is complete and orthogonal. This difficulty is aggravated by the "forbiddenness" of certain quantum numbers of the momentum of the system. We note that all these unpleasant circumstances naturally disappear in the weak-interaction limit or, equivalently, in the limit $V \to \infty$ ($V$ is the normalization volume). However, in order to study various concrete processes in the field of a quantized wave it is desirable to have a system of solutions free of these shortcomings at finite $V$ and to be able to take the limit $V \to \infty$ only in the final equations for the probabilities.

Below we find the Green's function for an electron in the field of a quantized wave; this Green's function is necessary for taking into account the interaction with other fields in second and higher orders of perturbation theory. We consider the simple case of a monochromatic quantized wave. To some extent, this calculation of the Green's function reveals the meaning of the "forbidden" zone. Finally, the Green's function can be used to point out a method for obtaining a complete, orthonormal system of wave functions for an electron in the field of a quantized wave which satisfies the natural asymptotic initial conditions: the limit $t \to \infty$ these functions become equal to the wave functions of the free electron.

Following Berson [1] we work from the equation for the Green's function $G(x', x)$ for the system consisting of the electron and the quantized wave:

$$G = \delta(x' - x) \delta(\xi' - \xi).$$

Here $\xi$ is the field variable in the "coordinate" representation [3] corresponding to a wave of frequency $\omega$ with polarization vector $e = (1, 0, 0, 0)$ and the photon 4-momentum $k = (0, 0, \omega, i\omega)$, $e = e_{\mu} \gamma_{\mu}$, and $\kappa x = \omega(x_3 - x_0)$. The Green's function transformation

$$O(x', x) = U_1(x, \xi) U_1^+(x', \xi') \tilde{G}(x', x'),$$

$$U_1(x, \xi) = \exp \left[ i \frac{\kappa x}{2} \left( \frac{\partial^2}{\partial \xi^2} - \frac{1}{\xi^2} \right) \right]$$
leads to an equation with constant coefficients, so that we can seek its solution as a fourfold Fourier expansion in the difference $x - x'$:

$$
\tilde{G}(xx', xx') = \int d^4q e^{i(q(x-x'))} \tilde{F}_q(t, \xi').
$$

(3)

The equation for the expansion coefficients $\tilde{F}_q$ is diagonalized by means of the transformation

$$
\tilde{F}_q(t, \xi') = U_2(q, \xi) F_q(t, \xi') U_2^{-1}(q, \xi), \quad U_2(q, \xi) = 1 + \frac{\kappa e^2}{2 q \kappa_0 V}
$$

(4)

which separates the spin and field variables,

$$
\left\{ \begin{array}{l}
\hat{A} + m + i \frac{\kappa}{2} \left[ \frac{\partial^2}{\partial^2 \xi} - \left( 1 - \frac{e^2}{q \kappa_0 V} \right) \left( \frac{e_1}{q \kappa_0 V} \right) \right] t + \frac{e^2 q_1}{(q \kappa)^3} \left( 1 - \frac{e^2}{q \kappa_0 V} \right) \right\} F_q(t, \xi') = \frac{\delta(t - \xi')}{(2\pi)^4}.
\end{array} \right.
$$

(5)

Now the problem of finding the Green's function can be reduced to the eigenvalue problem for the purely field operator in brackets in Eq. (5):

$$
\left[ \frac{\partial^2}{\partial^2 \xi} - \left( 1 - \frac{e^2}{q \kappa_0 V} \right) \left( \frac{e_1}{q \kappa_0 V} \right) \right] t + \frac{e^2 q_1}{(q \kappa)^3} \left( 1 - \frac{e^2}{q \kappa_0 V} \right) \right\} F_q(t, \xi') = \delta(t - \xi')
$$

(6)

where $f_\xi(t')$ are the eigenfunctions (which do not increase at $t \to \pm \infty$) corresponding to the eigenvalue $\varepsilon$.

If the completeness condition is satisfied for the system of functions $f_\xi$, the solution of Eq. (5) can be written

$$
F_q(t, \xi') = \frac{1}{(2\pi)^4} \int d\xi \left( \xi \xi' + \frac{\kappa}{2} \varepsilon \right) f_{\xi}(t) f_{\xi}(t').
$$

(7)

The complete Fourier integral in (3) includes the contribution from both the "allowed" and forbidden" zones, which are defined by the conditions $1 - e^2/q \kappa_0 V > 0$ and $1 - e^2/q \kappa_0 V < 0$, respectively. The solutions of the eigenvalue problem in these two cases, however, turn out to be quite different.

In the case $1 - e^2/q \kappa_0 V > 0$ the variable substitution

$$
\xi + \frac{e_1}{q \kappa_0 V} \left( 1 - \frac{e^2}{q \kappa_0 V} \right)^{-1/4}
$$

transforms (6) into the harmonic-oscillator equation, so that we find expressions for the eigenvalues $\varepsilon$, the eigenfunctions $f_\xi$, and the completeness condition:

$$
\xi = - (2n + 1) \left( 1 - \frac{e^2}{q \kappa_0 V} \right)^{1/2} + \frac{e^2 q_1}{(q \kappa)^3} \left( 1 - \frac{e^2}{q \kappa_0 V} \right)^{1/4}
$$

(8)

$$
\sum_{n=0}^{\infty} f_{\xi}(t) f_{\xi}(t') = \left( 1 - \frac{e^2}{q \kappa_0 V} \right)^{1/4} \delta(\xi - \xi').
$$

The Fourier transformation of the Green's function $F_q$ in this range of the variables $q$ is given by Eq. (7), where $\int d\xi$ should be replaced by the sum $(1 - e^2/q \kappa_0 V)^{-1/4} \sum_{n=0}^{\infty}$.

In the "forbidden" zone we have $1 - e^2/q \kappa_0 V < 0$, and we can use the variable substitution

$$
\xi + \frac{e_1}{q \kappa_0 V} \left( 1 - \frac{e^2}{q \kappa_0 V} \right)^{-1/4}
$$

then Eq. (6) transforms into a Schrödinger equation with an "inverted-operator" potential energy: