ON A CLASS OF STABLE "SPHERICAL" CONFIGURATIONS
IN THE KERR FIELD

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It is well known that among the plane ($\theta = \text{const}$) circular ($r = \text{const}$) stable motions of a test particle in the Schwarzschild field there is one which has a smallest value of the radius. A similar (lower) bound for equatorial circular orbits exists in the Kerr field (for $\theta = \pi/2$, $a = \frac{r_g}{2} - \varepsilon$, $0 < \varepsilon \ll r_g$) [2, 3]. In the present paper, a study is made of the possibility of obtaining similar results in the case of "spherical" motion ($r = \text{const}$, $\theta \neq \text{const}$) in the Kerr field with no a priori bound on the value of the parameter $a$.

Stable motions in the Kerr field with constant $r$ must satisfy the relations [1]

\begin{align}
U(r) &= E; \quad (1) \\
U'(r) &= 0; \quad (2) \\
U''(r) &\geq 0, \quad (3)
\end{align}

where the effective potential energy $U(\xi)$ is determined by the equation

\[ [U' \cdot (\xi^2 + a^2) - aL] - (\xi^2 + K)\Delta = 0, \quad (4) \]

and $U'$ and $U''$ are given by the expressions

\begin{align}
U' &= \frac{\partial U(\xi)}{\partial \xi} \bigg|_{r = \text{const}, \ L = \text{const}, \ K = \text{const}}, \\
U'' &= \frac{\partial^2 U(\xi)}{\partial \xi^2} \bigg|_{r = \text{const}, \ L = \text{const}, \ K = \text{const}}.
\end{align}

In these expressions and below, we use a system of units in which $c = 1$, $r_g = 1$, $m = 1$. The notation corresponds to [1].

The system of equations (1) and (2) determines a two-parameter family of "spherical" configurations. We denote it by $N'(a)$. Only two of the integrals of the motion, for example, $K$ and $r$, among the four integrals $E$, $K$, $L$, and $r$ which determine this motion are independent. The energy $E$ and the projection $L$ onto the symmetry axis can be represented in this case as functions of the two variables $K$ and $r$. For example,

\[ E^2 = \frac{(2r\Delta + (K + r^2)\Delta')^2}{16r^2(r^2 + K)\Delta}. \]

For the second derivative of the effective potential energy of the configurations $N'(a)$ we obtain

\[ U'' = \frac{2E}{r^2 + a^2} \cdot \frac{(K + r^2)[2\Delta - r(\Delta')^2] + (K + r^2)4r^2\Delta' - 4r^3\Delta^2}{(K + r^2)^32\Delta' + (K + r^2)4\Delta^2}. \]

It is important that the functional dependences determine the energy $E$, the angular momentum $L$, and $U''$ only up to the sign. This is a natural consequence of the circumstance that the Hamilton-Jacobi equation is a differential equation of second order. In principle, the family $N'(a)$ is defined for all values of $a$. This family defines the set of stationary configurations.

To separate from it stable configurations, it is necessary to use the condition (3). Depending on the values of the independent parameters $K$ and $r$, there are two possibilities: $U'' > 0$ or $U'' = 0$. In the first case, the configurations are stable. In the second case (it is considered in [1]), we obtain the system of equations $U = E$, $U' = 0$, $U'' = 0$, which, however, is insufficient to determine the stable configurations. Additional conditions are required. In [1], the stable configurations are determined by the conditions $U(r) = E$, $U' = 0$, $U'' = 0$, $U^{(IV)} > 0$. 

Stable configurations (we have denoted them by $M(a)$) satisfying these conditions do not have independent integrals of the motion. They are all functions of the parameter $a$:

$$E_0 = \frac{4r_0 - 1}{4r_0}, \quad L_0^2 = \frac{6r_0^2a^3 - r_0^4 - a^4}{4r_0a^2},$$

$$E_0L_0 = \frac{-2r_0^3 + 3r_0^3 + 2a^2r_0 - a^2}{4ar_0}, \quad K_0 = r_0^2, \quad b^2 = a^3 - 1/4;$$

Moreover, they are possible only for $a \geq \frac{1}{2}$. It can be seen from (8)-(10) that the sign of $E$ and $L$ of the configurations $M$ remains undetermined. Nevertheless, it is shown in [1] that the signs of these quantities cannot be chosen arbitrarily. And (from the logical point of view) only configurations $M_1$ and $M_2$ satisfying the following conditions are real:

$$E_{01} = \frac{1}{2} = E_{02} = \frac{1}{2}$$

Note that the configurations $M_1$ and $M_2$ satisfy the simple relation $K = r_0^2$. Thus, for fixed $a$ it corresponds to only two configurations $M(a)$.

Below, we shall consider in the class $N'(a)$ all the stable "spherical" configurations with the relation

$$K = r^2$$

between $K$ and $r^2$ which satisfy the "limiting" condition

$$E(a, r) = E_0(a) \text{ at } r = r_{01}(a), \quad E(a, r) = E_0(a) \text{ at } r = r_{02}(a).$$

These configurations thus form the set $N(a)$: $M_1(a), M_2(a) \subset N(a) \subset N'(a)$. It is clear that because of the additional condition (13) the set $\hat{N}(a)$ is defined only in the region $a \geq \frac{1}{2}$. The independent parameter (for fixed $a$) that determines the configurations of the set $N(a)$ is $r$.

Substituting (12) in the expression for $L^2$, and also in (6) and (7) for $E^2$ and $U''$, we obtain the following functions of $r$:

$$E^2 = \frac{f_2^2}{8r^2\Delta}, \quad L^2 = \frac{4r^2\Delta - (r^2 + a^2)f_2^2}{8r^2a^2\Delta},$$

$$U'' = \frac{-E \cdot f_1}{f_2 (r^2 + a^2) \Delta},$$

where

$$f_1 = 3r^3 - 2r + a^3, \quad f_2 = r^4 - 2r^2 + 2r(1 - a^2) - 2ra^2 + a^4, \quad \Delta = r^2 - r + a^2.$$

To determine the range of $r$ values of the configurations $N(a)$, we must plot the graph of $U''$ and find the values of $r$ at which $U'' > 0$. The expression (15) is not suitable for this purpose, since the sign of $E$ and, therefore, of $U''$ is not determined. The uncertainty in the signs of these quantities can be eliminated if the condition (13) is used. Let us show this.