ON A CLASS OF STABLE "SPHERICAL" CONFIGURATIONS
IN THE KERR FIELD

S. V. Izmailov and E. S. Levin

It is well known that among the plane (θ = const) circular (r = const) stable motions of a test particle in the Schwarzschild field there is one which has a smallest value of the radius. A similar (lower) bound for equatorial circular orbits exists in the Kerr field (for θ = π/2, a = rg/2 - ε, 0 < ε ≪ rg) [2, 3]. In the present paper, a study is made of the possibility of obtaining similar results in the case of "spherical" motion (r = const, θ ≠ const) in the Kerr field with no a priori bound on the value of the parameter a.

Stable motions in the Kerr field with constant r must satisfy the relations [1]

\[ U(r) = E; \]
\[ U'(r) = 0; \]
\[ U''(r) > 0, \]

where the effective potential energy \( U(\xi) \) is determined by the equation

\[ [U : (\xi^2 + a^2)L - (\xi^2 + K)\Delta] = 0, \]

and \( U' \) and \( U'' \) are given by the expressions

\[ U' \equiv \frac{\partial U(\xi)}{\partial \xi} \bigg|_{\xi = r, L = \text{const}, K = \text{const}}, \quad U'' \equiv \frac{\partial^2 U(\xi)}{\partial \xi^2} \bigg|_{\xi = r, L = \text{const}, K = \text{const}} \]

In these expressions and below, we use a system of units in which \( c = 1, \; rg = 1, \; m = 1 \). The notation corresponds to [1].

The system of equations (1) and (2) determines a two-parameter family of "spherical" configurations. We denote it by \( N'(a) \). Only two of the integrals of the motion, for example, \( K \) and \( r \), among the four integrals \( E, K, L, \) and \( r \) which determine this motion are independent. The energy \( E \) and the projection \( L \) onto the symmetry axis can be represented in this case as functions of the two variables \( K \) and \( r \). For example,

\[ E^2 = \frac{[2r\Delta + (K + r^2)\Delta']^2}{16r^2(r^2 + K)\Delta}. \]

For the second derivative of the effective potential energy of the configurations \( N'(a) \) we obtain

\[ U'' = \frac{2E}{r^2 + a^2} \frac{(K + r^2)[2\Delta - (r(\Delta')^2] + (K + r^2)4r^2\Delta' - 4r^2\Delta^2}{(K + r^2)^2[2\Delta + (K + r^2)\Delta'] + (K + r^2)4r^2\Delta' + (K + r^2)4r\Delta^2}. \]

It is important that the functional dependences determine the energy \( E \), the angular momentum \( L \), and \( U'' \) only up to the sign. This is a natural consequence of the circumstance that the Hamilton–Jacobi equation is a differential equation of second order. In principle, the family \( N'(a) \) is defined for all values of \( a \). This family defines the set of stationary configurations.

To separate from it stable configurations, it is necessary to use the condition (3). Depending on the values of the independent parameters \( K \) and \( r \), there are two possibilities: \( U'' > 0 \) or \( U'' = 0 \). In the first case, the configurations are stable. In the second case (it is considered in [1]), we obtain the system of equations \( U = E, \; U' = 0, \; U'' = 0, \) which, however, is insufficient to determine the stable configurations. Additional conditions are required. In [1], the stable configurations are determined by the conditions \( U(r) = E, \; U' = 0, \; U'' = 0, \; U'''' > 0 \).


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Stable configurations (we have denoted them by $M(a)$) satisfying these conditions do not have independent integrals of the motion. They are all functions of the parameter $a$:

\begin{align*}
E_0 &= \frac{4r_0 - 1}{4r_0}, \quad L_0 = \frac{6r_0^3 a^3 - r_0^2 - a^4}{4r_0 a^2}, \\
E_0 L_0 &= \frac{-2r_0^2 + 3r_0^2 + 2a^2 r_0 - a^2}{4a r_0}, \quad K_0 = r_0^2, \quad b^2 = a^3 - 1/4; \\
r_0 &= \{r_{01}, r_{02}\}, \quad r_{01} = 2^{-1} + b + \sqrt{b}, \quad r_{02} = 2^{-1} + b - \sqrt{b}.
\end{align*}

Moreover, they are possible only for $a \geq \frac{1}{2}$. It can be seen from (8)-(10) that the sign of $E$ and $L$ of the configurations $M$ remains undetermined. Nevertheless, it is shown in [1] that the signs of these quantities cannot be chosen arbitrarily. And (from the logical point of view) only configurations $M^+$ and $M^-$ satisfying the following conditions are real:

\begin{align*}
E_01 \big|_{a=1/2} &= -E_02 \big|_{a=1/2} = 1/\sqrt{2}.
\end{align*}

Note that the configurations $M^+$ and $M^-$ satisfy the simple relation $K_0 = r_0^2$. Thus, for fixed $a$ it corresponds to only two configurations $M(a)$.

Below, we shall consider in the class $N'(a)$ all the stable "spherical" configurations with the relation

\begin{align*}
K = r^2
\end{align*}

between $K$ and $r^2$ which satisfy the "limiting" condition

\begin{align*}
E(a, r) = E_01(a) \text{ at } r = r_{01}(a), \quad E(a, r) = E_02(a) \text{ at } r = r_{02}(a).
\end{align*}

These configurations thus form the set $N(a)$: $M^+_1(a), \quad M^-_1(a) \subset N(a) \subset N'(a)$. It is clear that because of the additional condition (13) the set $\tilde{N}(a)$ is defined only in the region $a \geq \frac{1}{2}$. The independent parameter (for fixed $a$) that determines the configurations of the set $N(a)$ is $r$.

Substituting (12) in the expression for $L^2$, and also in (6) and (7) for $E^2$ and $U''$, we obtain the following functions of $r$:

\begin{align*}
E^2 &= \frac{f_2^2}{8r^2 \Delta}, \quad L^2 = \frac{[4r^2 \Delta - (r^2 + a^2) f_1]^2}{8r^2 a^2 \Delta}, \\
U'' &= \frac{-E f_1}{f_2 (r^2 + a^2) \Delta},
\end{align*}

where

\begin{align*}
f_2 &\equiv 3r^2 - 2r + a^2, \quad f_1 \equiv r^4 - 2r^3 + 2r (1 - a^2) - 2ra^2 + a^4, \quad \Delta \equiv r^2 - r + a^2.
\end{align*}

To determine the range of $r$ values of the configurations $N(a)$, we must plot the graph of $U''$ and find the values of $r$ at which $U'' > 0$. The expression (15) is not suitable for this purpose, since the sign of $E$ and, therefore, of $U''$ is not determined. The uncertainty in the signs of these quantities can be eliminated if the condition (13) is used. Let us show this.