A solution to the equations of the relative dynamics of test bodies in the framework of general relativity is constructed for the case of slowly varying coefficients (when these last depend on the "truncated" proper time on a reference trajectory). Nonlinear radial oscillations of a test body in the neighborhood of a point of equilibrium in the Nordström space are considered. Some nonlinear effects of stochastic forces in the relative dynamics of test bodies are considered in a space of constant curvature.

1. Single-Frequency Oscillations in the Relative Dynamics of Test Bodies in General Relativity in the Case of Slowly Varying Parameters

In the preceding parts [1, 2] of the present paper, we investigated nonlinear resonance phenomena in the relative dynamics of test bodies in general relativity for stationary geodesics. In this case, the coefficients of the system \((l_{I}, 2_{I})\) were constants on the reference trajectories and the principal directions of the matrix \(K_{n}^{m}\) were obtained by parallel transport along \(\Gamma_{0}\). It is clear that such an assumption, although having important applications, significantly reduces the class of motions for which the procedure for constructing the asymptotic series for nonlinear single-frequency oscillations is valid. In this connection, we here pose the problem of constructing analogous solutions for the special but important case when the coefficients of the system \((l_{I}, 2_{I})\) depend on the "slow" proper time \(\tau = \varepsilon s\), where \(\varepsilon\) is a small parameter, i.e., when the coefficients of the system vary slowly along the reference geodesic. Note that on the transition to differentiation with respect to the new parameter \(\tau\) new terms do not occur in the system \((l_{I}, 2_{I})\), since the parameter is again canonical (\(\varepsilon = \text{const}\)). Although this case is also fairly special, it nevertheless significantly extends the possibilities of the method, since this class includes the majority of problems of applied nature. For example, the study of single-frequency oscillations in the relative dynamics of test bodies reduces to this case when the reference trajectory is a trajectory close to a trajectory of the group of motion of the space \(V_{n}\); in particular, this is so in centrally symmetric fields in the case when the reference motion has small eccentricity.

Below, we construct an asymptotic series to accuracy \(\varepsilon^{2}\) for the system \((5_{I})\) under the assumption that its coefficients vary slowly along \(\Gamma_{0}\). We give here the construction only for the case of principal resonance, since the construction is analogous for an arbitrary resonance.

We consider the solution of the system \((5_{I})\) on a finite interval of the proper time \(0 \leq s \leq T\), where \(T = L/\varepsilon\), and for given small \(\varepsilon\) the value of \(L\) can be arbitrarily large. One of the most important features of the system \((5_{I})\) is that in the given case the principal directions \(l_{i} (i = 1, 2, 3)\) of the matrix

\[
K_{\phi}^{0} = R_{3}^{0} U_{3} U_{3} U_{3}^{*}
\]

do not undergo parallel transport along \(\Gamma_{0}\). Therefore, if the system \((l_{I}, 2_{I})\) is written down in an orthogonal frame and the matrix \((1)\) given diagonal form, this frame

will undergo some transport (not parallel) along \( \Gamma_0 \),
\[
\frac{D}{d\tau} L^\mu = L^\mu, 
\]
where \( L_{\mu t} = -L_{tm} \) is the projection of the Ricci coefficients onto \( \Gamma_0 \). For given matrix \( K_{\mu}(s) \), it is not difficult to determine these coefficients. We denote by \( \omega(t) \) the eigenvalues of the matrix \( K_{\mu}(s) \), i.e., \( \omega(t) \) are the solution to the equation
\[
\text{det} \left[ K_{\mu}(s) - \omega(t) L^\mu \right] = 0, 
\]
in which we regard \( s \) as a parameter. We assume that for all \( s \in [0, T] \) all the \( \omega(t) \) are different (i.e., there is no interior resonance). Let \( L^\mu_i \) \( (i = 1, 2, 3) \) be the principal directions of the given matrix corresponding to the eigenvalues \( \omega(t) \). If we differentiate covariantly along \( \Gamma_0 \) the relationship that determines the principal directions of the matrix,
\[
(K_{\mu}(s) - \omega(t) L^\mu_i) L^\mu_i = 0 \quad (i = 1, 2, 3), 
\]
and then project these relations onto \( L^\mu_i \), we obtain from the relations (2)
\[
L^\mu_i (s) = \frac{D}{d\tau} K_{\mu}(s) L^\mu_i - \frac{D}{d\tau} K_{\mu}(s) L^\mu_i \frac{\partial}{\partial t} \frac{\partial}{\partial t} 
\]
(no summation over \( t \)). Since in the considered case the matrix \( K_{\mu} \) depends on the "slow"
proper time \( \tau \), for the rotation coefficients we obtain
\[
L^\mu_i (\tau) = \frac{D}{d\tau} K_{\mu}(\tau) L^\mu_i - \frac{D}{d\tau} K_{\mu}(\tau) L^\mu_i \frac{\partial}{\partial t} \frac{\partial}{\partial t} 
\]
where
\[
K_{\mu}(\tau) = \frac{D}{d\tau} K_{\mu}(\tau) L^\mu_i \frac{\partial}{\partial t} \frac{\partial}{\partial t} 
\]
The rotation coefficients have the order \( \epsilon \), and, therefore, in the considered case the relative dynamics of the test bodies is described by the system of equations (1_i, 2_i), for which the coefficients are determined in accordance with (1_i), (2_i), and (6).

We consider the linear system (unperturbed motion)
\[
\xi^\mu + K_{\mu}(\tau) \xi^\tau = 0. 
\]
Suppose that it admits undamped harmonic two-parameter oscillations corresponding to the frequency \( \omega \), and also that the unique point of equilibrium is the trivial solution \( \xi^\mu = 0 \).

In accordance with [3], we seek the solution to the system (1_i, 2_i) for the case of the principal resonance in the form of the series
\[
\xi^\mu_i (s) = \xi^\mu_i (\tau) t \cos \psi + \sum \epsilon^\mu U^\mu_i (\tau, \psi, \psi, \phi, \psi), 
\]
the amplitude and phase being determined by the relations
\[
a_t = \sum_{n=1}^{\infty} \epsilon^n A_n (\tau, \psi, \psi); \quad b_t = \omega (\tau) = \psi + \sum \epsilon^n B_n (\tau, \psi, \psi). 
\]
Since our aim is to construct a solution to accuracy \( \epsilon^2 \), we must determine the coefficients \( U^\mu_i, A_1, B_1, A_2, B_2 \). To determine these, we must make lengthy calculations, which we shall not reproduce here but indicate only the main points and give the final results. The basic idea used in these calculations consists of substituting (8), differentiated by virtue of (9) in the original system (1_i), the coefficients of corresponding powers of \( \epsilon \) then being equated [3]. The correction coefficients \( U^\mu_i \) can be determined from the