Vector fields $\xi_i$, corresponding to the Poincaré group generators (infinitesimal translations and rotations) are defined by first-order differential conditions. These equations have nontrivial solutions in an arbitrary torsionless Riemannian space, and can be considered as a generalization of the definition of translations and rotations in flat space. The equations for translations can be integrated.

For a space with the Minkowski topology, if the boundary conditions at infinity are shown so that the space is asymptotically flat, the solution is unique. The vector fields $\xi_i$ specify a physical system as a whole.

It is known that a flat Minkowski space-time is invariant under motions, i.e., translations and rotations. This has fundamental significance for physics [1-4]. Motions are defined so that they preserve local invariants [1, 5-8]. Under such definition, an arbitrary curved Riemannian space does not allow motion.

Below we shall give another definition of motion, and prove the existence of translations and rotations which are close to unity in an arbitrary Riemannian space, thus defining the Poincaré group generators. Note that for such gravitational fields which have asymptotically flat metric at the space infinity, the conservation laws allow to define the Poincaré group there [9]. The proof of [9] makes use of the specific form of the Lagrangian of the gravitational field. Here we shall show that the Poincaré group can be defined from purely geometric considerations, and in the entire space.

1. In a flat space $R_{ikjm} = 0$, and it is always possible to introduce a Cartesian metric with constant coefficients:

$$ds^2 = g_{ik}dx^idx^j; \quad \frac{\partial}{\partial x^2}g_{ik} = 0, \quad \sqrt{-g} = 1. \tag{1}$$

Group generators correspond to infinitesimal transformations:

$$x^i \rightarrow x^i + \xi^i. \tag{2}$$

For a translation we have:

$$\xi^i = \text{const}, \quad \partial_m \xi^i = 0, \quad \partial_m \frac{\partial}{\partial x^m} = 0, \tag{3a}$$

and for a rotation

$$\xi_i = \frac{1}{2} e_{iklm} x^k \xi^l m, \quad \partial_m \xi^i = 0, \tag{3b}$$

where $e_{ikjm}$ is the unit entirely antisymmetric tensor, and $\mu^i m$ — the bivector defining the rotation. At some point $M_0$ the parameters defining the translation or rotation must be given:

$$\xi^i (M_0); \quad \mu^i m (M_0), \tag{4}$$

and since these are constant, defining them at one point defines them in the entire space-time. Note that the translation (rotation) is defined everywhere, including such areas which are not connected to $M_0$ by a timelike curve leading to the future from $M_0$. Since the field is defined in the entire space, in case of a more complicated geometry it can be used to study global, rather than local, properties of space-time [8], thus completing the methods.
based on study of families of timelike geodesic lines [8]. In order to do that, Eqs. (2) and (3), defining the rotation or translation, have to be written in an arbitrary Riemannian space, and the existence of solutions must be proved. This will be done below.

2. The definitions of a translation or rotation are written in a Cartesian coordinate system. In order to go over to arbitrary coordinates, they have to be written as differential conditions, substituting common derivatives by the covariant ones. The definition of a translation will then be:

\[ \nabla_{m} \xi^{i} = 0 \quad 16 \text{ conditions,} \]

where the number of conditions is given for the four-dimensional space. Similarly, for a rotation we have

\[ \nabla_{m} \Gamma^{lm} = 0 \quad 24 \text{ conditions,} \]

\[ \nabla_{r} \xi^{i} - \nabla_{i} \xi^{r} = 0, \quad \nabla^{2} \xi^{i} = 0. \]

Note that if we limit ourselves to first order differential conditions, Eq. (5) is uniquely defined. Integrating Eq. (5), in internal geometry we can construct the fields \( \xi_{1} \), corresponding to translations or rotations, in arbitrary coordinate system.

3. Consider now an arbitrary Riemannian space with the metric

\[ ds^{2} = g_{ij}(x^{r})d x^{i} d x^{j}, \]

in torsionless spaces: \( \nabla_{[i} \xi_{j]} = 0 \). The difficulty is that in arbitrary Riemannian space, Eqs. (5) can be incompatible, and therefore have only zero solutions. This is related to the fact that covariant derivatives do not commute:

\[ (\nabla_{i}\nabla_{j} - \nabla_{j}\nabla_{i}) \xi^{k} = -\xi^{m} R_{mkij}. \]

If Eq. (5a) holds, \( \xi^{m} R_{mkij} = 0 \) for any vector of the full basis \( \xi_{(a)} \), so that nontrivial solutions are possible only in exceptional cases. Let us split the conditions (5a) into three types:

\[ \nabla_{i} \xi_{k} + \nabla_{k} \xi_{i} = 0 \quad 19 \text{ conditions,} \]

\[ \nabla_{i} \xi_{k} - \nabla_{k} \xi_{i} = 0 \quad 6 \text{ conditions,} \]

\[ \nabla^{2} \xi_{i} = 0 \quad 1 \text{ condition.} \]

Note that (8a) includes (8c). Equations (8a) are called Killing conditions, and can be entirely integrated only in a space of constant curvature [10], in particular, in a flat space.

In this paper we shall substitute the conditions (5) by less restricting ones, in order to be able to prove the existence and uniqueness of solutions in arbitrary Riemannian spaces.

4. First consider the case of translations [11]. Since the condition (8a) cannot hold, let us drop it, retaining the remaining seven conditions (8b), (8c), which will define the desired translation field. The problem is reduced to integrating a scalar wave equation

\[ \Box \varphi = 0, \Box = \nabla_{i} \nabla^{i}, \]

and therefore the existence of the solution is proved. This procedure of constructing a vector field of translation was called a "harmonic" transfer by V. A. Fok [11]. Unlike the parallel transfer, the harmonic transfer does not depend on the path and does not conserve the scalar product.

5. Now consider rotations. The 24 conditions (5b) for \( \mu^{lm} \) also are too rigid. From Eq. (5b) we can see that the antisymmetric tensor \( \mu^{lm} \) allows the existence of a vector potential \( \xi_{1} \), in full analogy with the equations of electromagnetic field in a vacuum:

\[ \xi_{i} \leftrightarrow A_{i}; \mu_{ik} \leftrightarrow F_{ik}. \]

Therefore, we can immediately write the desired less rigid equations:

\[ \nabla_{i} \mu_{j} + \nabla_{j} \mu_{i} + \nabla_{k} \xi_{kl} = 0 \quad (\text{cycle},) \]

\[ \nabla^{2} \mu_{ik} = 0. \]

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