DISPERSION OF GRAVITATIONAL WAVES
IN A RELATIVISTIC GAS

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The propagation of gravitational waves in a relativistic gas which is in a static gravitational field in the unperturbed state is considered. The group velocity of the transverse gravitational waves and the decrements of collisional and collisionless damping are calculated.

The linear equations for gravitational field perturbations in a medium are [1, 2]:

\[
\Delta_2 \psi^{\mu} = - \nabla^\mu \nabla_\mu \phi + 2R^{\mu}_{\nu} \phi + 2T^{\mu}_{\nu} \nabla_\nu - 2\kappa T^{\mu}_{\nu} - \frac{1}{2} \nabla^\mu \nabla_\nu T^{\nu}_{\mu} + \frac{1}{2} \nabla^\mu \nabla_\nu \phi T^{\nu}_{\mu} - xT^{\mu}_{\nu} x^{\nu} - xT^{\mu}_{\nu} x^{\nu} = - 2\kappa \phi,
\]

(1)

where \( \Delta_2 \) is the Beltrami operator of the second kind; \( T_{\mu\nu} \) is the energy–momentum tensor of the medium in the unperturbed state; \( T^{\mu}_{\nu} = \tilde{T}_{\mu\nu} - T_{\mu\nu} \) is the perturbation of the energy–momentum tensor. Let us use the energy–momentum tensor of a multicomponent relativistic gas [3]

\[
T^{\mu}_{\nu}(x) = \sum_a T^{\mu}_{\nu}_{a}(x) = \sum_a \int dP f_a(x, p) p^\mu p^\nu
\]

(2)
as the energy–momentum tensor (the summation is over all species of particles). The distribution function of each gas component \( f_a(x, p) \) satisfies the kinetic equation [3]

\[
\left\{ p^\mu \frac{\partial}{\partial x^\mu} - T_{\nu}^{\mu}(x) p^\mu p^\nu \frac{\partial}{\partial p^\nu} \right\} f_a(x, p) = \sum_b I_{ab}(x, p);
\]

(3)

where \( I_{ab}(x, p) \) is the collision integral between particles of species \( "a" \) and particles of species \( "b" \). Let the unperturbed space be static, i.e., admit the timelike Killing vector \( \xi^i(x) \). Then the distribution function corresponding to the equilibrium state of the gas is [4]

\[
f^2_a(x, p) = A_a \exp \left\{ - (\xi^i p_i) \right\},
\]

(4)

(\( A_a \) is a normalizing constant). The vector \( \xi^i(x) \) is normalized in such a way that

\[
\nu_a(x) = m_a c (\xi^i, \xi^i)^{1/2} = \frac{m_a c^2}{\Theta_a(x)}
\]

where \( \Theta_a(x) \) is the local temperature of the \( a \)-th component, \( u_a^i(x) = \xi_a^i / (\xi^i, \xi^i)^{1/2} \) is the local velocity of macroscopic motion of this component. By virtue of the 4-momentum normalization relationship, the component \( p^4 \) is renormalized in the perturbed state:

\[
p^4 = p^4 - \frac{1}{2} \frac{h_{kk} p^k p^k}{p^4} + \ldots
\]

Consequently, the perturbed distribution function \( f_a(x, p') \) should be selected in the form

\[
f_a(x, p') = f^2_a(x, p') + \delta f_a(x, p'),
\]

(5)

where \( \delta f_a \) is the deviation of the distribution function from equilibrium, and \( p' = (p^1, p^2, p^3, p^4) \). Substituting the function (5) into (3), we obtain in the linear approximation

\[ \left\{ p^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}(x) p^j p^k \frac{\partial}{\partial p^i} \right\} \delta f_a(x, p) = - \Omega_{ij} \cdot p^i p^j \overline{f}_a(x, p) + \sum_b \Gamma_{ab}(x, p). \] (6)

In this approximation the collision integral depends linearly on \( \delta f_a(x, p) \) and \( \delta \tilde{f}_a(x, p) \). Moreover, let us calculate the perturbed energy–momentum tensor \( T_{ik}^{\text{pert}} \) by using (2) for the function (4). An explicit form of the energy–momentum tensor of the equilibrium state [4], as well as differential relationships between the Bessel functions, must be used for the calculations. As a result of these calculations, the equations (1) become

\[ \Delta_k \phi^{ik} - \phi^{ik} \phi^{j lk} + 2R^{ik}_{j lk} \phi^{lm} + 4\pi \sum_a A_a(m_a c)^i \]

\[ \times \left\{ \left( \psi_{sa} - \frac{1}{2} \right) \left[ K_i \left( \psi_{sa} - K_p \left( \psi_{sa} \right) \left( \xi_a + \xi_a \right) \right) \right] - K_p \left( \psi_{sa} \right) \right\} = - 2 \sum \xi^{ik}(x), \] (7)

where \( \psi_{sa} = \psi^{ik}_{sa} \cdot u_{sa} \cdot g_{sa} \), \( K_p(\mu) \) are modified Bessel functions

\[ \xi^{ik}(x) = \int d\psi f_a(x, p) p^i p^k. \] (8)

The system of equations (6), (7) in combination with the definition (8) is the desired system of linear equations describing the propagation of weak gravitational waves in a relativistic gas in a static gravitational field \( g_{ik}(x) \).

Furthermore, let us limit ourselves to the consideration of sufficiently short gravitational waves whose length is small compared with the background inhomogeneity

\[ u_a \simeq c \frac{\partial \xi_{ik}}{\partial x} \ll \omega. \] (9)

In this case, a significant simplification of the problem is achieved by representing the perturbed quantities as

\[ \psi^{ik}(x) \rightarrow \psi^{ik} \cdot \exp \left[ i V(x) \right], \] (10)

where \( V(x) \) is an eiconal. By using the eiconal, the wave vector of the oscillations \( k_l \) can be introduced

\[ \kappa_i(x) = \partial_i V(x). \] (11)

Substitution of \( \psi^{ik} \) in the form (10) into (6)–(7) yields in a first approximation in the parameter \( \omega_g/\omega \)

\[ -(\kappa, \kappa) \psi^{ik} + \kappa_i (\kappa^i \psi^{ik} + \kappa^k \psi^{ik}) + 2R^{ik}_{j lk} \psi^{lm} + 4\pi \sum_a A_a(m_a c)^i \]

\[ \times \left\{ \left( \psi_{sa} - \frac{1}{2} \right) \left[ K_i \left( \psi_{sa} \right) \right] - K_p \left( \psi_{sa} \right) \right\} = - 2 \sum \xi^{ik}; \] (12)

\[ i (k, p) \delta f_a - \Gamma_{ik} p^i p^k \frac{\partial f_a}{\partial p^i} = - i h_{ik} p^i p^k \delta f_a + \frac{i}{2} h_{ik} p^i p^k \cdot (k, p) f_a^0 + \sum_b \Gamma_{ab}. \] (13)

As is seen from (13), the distribution function \( \delta f_a \) in the collisionless approximation consists of two parts: the first is nonresonant and the second is resonant, with a singularity at the point

\[ (k, p) = 0. \] (14)

The presence of this singularity results in the appearance of gravitational wave damping in the collisionless approximation.

The importance of taking account of the gravitational background generated by the gas in the unperturbed state in solving problems about gravitational radiation propagation in a medium should be noted, even in case this background is vanishingly small. Indeed, it is easy to estimate the order of \( T_{ik}^{\text{pert}} \) from (7) and (8) in the collisionless approximation, but such, namely, is the order of magnitude of \( T_{ik}^{\text{pert}} \) associated with the presence of the background.

Now let us solve the problem of gravitational radiation propagation in a homogeneous gas of sufficiently low density by taking account of the background generated by the gas itself. In the collisionless approximation, the distribution function \( \delta f_a^{(1)} \) is in conformity with (13)