Since the emission spectrum has peaks at frequencies $\gamma = n\pi$, substituting $\gamma = n\pi$ into (17) we find that a spiral undulator of finite length permits generating circularly polarized radiation only at odd harmonics, i.e., with $\gamma = (2n + 1)\pi$.

LITERATURE CITED


INVERSE PROBLEM FOR THE SCHRODINGER EQUATION WITH A REPULSIVE POTENTIAL

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We study the dependence of the repulsive potentials on the discrete spectral characteristics of the Schrödinger equation. The behavior of the regular solutions and the corrections to the potential for various changes to the spectrum are analyzed. It is shown that for a change in the number of bound states, the asymptotic correction to the potential is related to the period of classical vibrations in the field of the reference potential.

Using the inverse problem in the quantum theory of scattering [1, 2], the dependence of arbitrary central field repulsive potentials (those potentials which are increasing at infinity) on the discrete spectral characteristics of the radial Schrödinger equation has been studied. This problem is of current interest in the study of bound states of quarks and antiquarks (families of heavy mesons), coherent states, and other problems. The construction of a potential with specified spectral characteristics is in itself of interest because the number of cases in which the Schrödinger equation can be solved exactly is very limited.

We use a system of units where $\hbar = 2m = 1$ and consider the radial Schrödinger equation $(0 \leq r < \infty)$:

$$
\frac{d^2}{dr^2} \varphi(E, r) + [E - V(r)] \varphi(E, r) = 0; \quad V(r) = V(r) + l(l + 1) \frac{1}{r^2}
$$

with a repulsive potential $V(r)$ which satisfies the conditions $\lim_{r \to 0} V(r) = 0$, $\lim_{r \to \infty} V(r) = \infty$, $V'(r) > 0$ for $r > r_0 \geq 0$ but is otherwise arbitrary. For a given value of the angular momentum $l$, Eq. (1) has a purely discrete spectrum with an infinite number of energy levels $E_{ln}$ where $n = 0, 1, 2, \ldots$, and where we assume that the energy increases with increasing $n$.

We study the change in the potential $V(r)$ (this is called the reference potential) and regular solutions $\varphi(E, r) \equiv r^{l+1}$ of the Schrödinger equation resulting from a change in the spectrum $E_{ln}, C_{ln} = \left(\int_0^\infty \varphi^2(E_{ln}, r) dr\right)^{-1}$ to new values $\tilde{E}_{ln}, \tilde{C}_{ln} (n = 0, 1, 2, \ldots)$. Although the repulsive reference potential $V(r)$ increases as $r \to \infty$, the new potential $\tilde{V}(r) = V(r) + \Delta V(r)$ and new regular solutions $\tilde{\varphi}(E, r)$ of the Schrödinger equation with this potential (as in the usual case of a short-range potential [1, 2]) can be obtained with the Gel'fand-Levitan method [3-5]. However, the resulting formal expressions for $\varphi(E, r)$ and $\tilde{V}(r)$ must be verified by substitution into the Schrödinger equation when the potential is repulsive. When the change
to the spectrum involves a finite number of elements $E_{\xi, n}$, $E_{\eta, n}$, the Gel'fand–Levitan equation can be solved in closed form. In this case three situations are possible, each of which corresponds to a definite form of the asymptotic correction to the potentials $\Delta V(r)$ for $r \to \infty$:

1. Change in the Number of States over a Finite Region of the Energy $e < E < E^0_J (e = \max \sqrt{V(r)}$ for $r < r_0)$ by an Amount $\Delta N = N_2 - N_1$. This corresponds to the removal of $N_1$ arbitrary states with energies $E_{\xi, i} = E_{\xi, n_1} (i = 1, 2, \ldots, N_1)$ and the addition of $N_2$ new states with energies $E_{\eta, i}$ and normalization constants $C_{\xi, i} (i = N_1 + 1, \ldots, N_1 + N_2) [3, 5]$.

2. Simultaneous Removal and Addition of an Identical Number of States. This corresponds to a shift of a finite number of levels with changes in their normalization constants $[4]$. This formally reduces to case (1) with $\Delta N = 0$, $N_1 = N_2 \neq 0$.

3. A Change in a Single Normalization Constant $C_{\xi, i} N_2$ of an Arbitrary State of the Schrödinger Equation without Changing the Number of States of Shifting the Energy Levels $[1, 2]$.

Since the method of inverse problems is well developed (see the literature cited in [2]) we present the results without showing the calculations for the formal (since $\lim_{r \to \infty} V(r) = 0$) regular solutions $\tilde{\varphi}(E, r)$ and correction to the potential $\Delta V(r)$ for a finite number $N = N_1 + N_2$ of changes to the elements of the spectrum $E_{\xi, n}$, $C_{\xi, n}$. This corresponds to a combination of cases (1)-(3):

$$\tilde{\varphi}(E, r) = \varphi(E, r) - \frac{1}{(E - E_i)} \sum_{i,j=1}^{N} A_{ij}(r) b_j \varphi(E_{ij}, r) \int_{0}^{r} \varphi(E_{ij}, r') \varphi(E, r') \, dr';$$

(2)

$$\Delta V(r) = -2 \frac{d^2}{dr^2} \ln \varphi(r); \quad \varphi(r) = \det a_{ij}(r);$$

(3)

$$a_{ij}(r) = b_{ij} + b_i \int_{0}^{r} \varphi(E_{ij}, r') \varphi(E_{ij}, r') \, dr'.$$

(4)

Where $A_{ij}(r)$ is the algebraic cofactor matrix of $a_{ij}(r)$, $b_i$ is the change in the normalization constant $(i = 1, 2, \ldots, N_1; i = N_1 + N_2 + 1, \ldots, N)$ or normalization constants $C_{\xi, i} (i = N_1 + 1, \ldots, N_1 + N_2)$ of the newly added state and the summation over $i, j = 1, 2, \ldots, N$ corresponds to all changes of the elements of the spectrum.

For physical applications, of most interest are the behavior of the regular eigenfunctions $\tilde{\varphi}(E_{\xi, n}, r)$ and the correction $\Delta V(r)$ in the limits $r \to 0$ and $r \to \infty$.

For small $r$ the regular eigenfunctions and the correction to the potential have the form

$$\tilde{\varphi}(E_{\xi, n}, r) \approx r^{l+1}, \quad \Delta V(r) \approx -4 (l + 1) r^{2l+1} \sum_{i=1}^{N} b_i + O(r^{2l+3}),$$

(5)

which for $l = 0, \sum_{i=1}^{N} b_i \neq 0$ corresponds to a significant distortion of the reference potential near the origin.

The asymptotic behavior ($R \to \infty$) of the regular eigenfunctions $\tilde{\varphi}(E_{\xi, n}, r)$ and the correction to the potential is more complicated. However, starting from the usual asymptotic form of the regular solutions $[6]$

$$\tilde{\varphi}(E, r) \sim [V_i(r) - E]^{-1/4} \exp \left\{ \pm \int_{r}^{R} [V_i(r') - E]^{1/2} \, dr' \right\}$$

(where the minus sign is for $E = E_{\xi, n}$ and the plus sign is for $E \neq E_{\xi, n}$, $n = 0, 1, 2, \ldots$ and $r$ is defined to be the largest root of the equation $V_i(r) = E$) an extremely compact expression can be obtained for $\tilde{\varphi}(E_{\xi, n}, r)$ and $\Delta V(r)$. For all three of the cases mentioned above, there is a similar asymptotic behavior of the new eigenfunctions $\tilde{\varphi}(E_{\eta, n}, r)$, $\kappa = 0, 1, 2, \ldots$ which is similar (same exponential decay) to the asymptotic behavior of the reference functions $\varphi(E_{\xi, n}, r)$, $n = 0, 1, 2, \ldots$. The correction to the potential $\Delta V(r)$ for the removal of $N_1$ states and addition of $N_2$ new states at large $r$ can be obtained from (3):