A generalization of the theory of algebraic properties is proposed for an equation of general form, providing a tool for the acquisition of new results with regard to the symmetry of certain nondifferential equations of theoretical physics.

In past investigations of systems of differential equations extensive use has been made of the theory of group properties [1, 2]. However, the equation that admits a nontrivial Lie group of point transformations of the space of independent and dependent variables is a rare exception. This fact is hardly surprising, considering that the corresponding symmetry group satisfies the following stringent requirement: The general element of the algebra of infinitesimal operators of the group is a first-order scalar differential operator. The latter condition accounts for the attempts to generalize the theory of group properties. It turns out that a valuable tool for the integration and qualitative analysis of a differential equation is the algebra of differential operators of order $\geq 1$ admitted in some definite sense by the equation [3]; what is significant here is that this algebra contains operators admitted by the equation in the sense of S. Lie. A broader generalization has been devised for a linear (not necessarily differential) equation of general form, viz., the algebra of linear symmetry operators [4]. One benefit of the latter conception is the possibility of algebraic classification of the solutions of equations in quantum field theory [5].

It would be most advantageous to generalize the theory of group properties in such a way as to embody all the generalizations cited above. Such a master generalization would afford the possibility of computing the admissible operators and formulating general properties of algebras of many different equations, all by a single unified scheme. In the present article we set forth one possible solution of the desideratum just stated. In order to endow the ensuing discussion with greater clarity we frame it in Banach spaces. The fundamental results can be translated to countably normed spaces without significant alterations.

Let there be given an infinitely (Fréchet-) differentiable mapping $F$ of Banach space $X(x)$ into Banach space $Y(y)$, where the kernel $K = \{x | F(x) = 0\}$ of the mapping contains only regular elements: $F'(x)(X) = Y$ for every $x \in K$.

Definition 1. $T: X \to X$ is a symmetry operator of the mapping $F$ [of the equation $F(x) = 0$] if $T(K) \subseteq K$.

Clearly, the set of symmetry operators is nonempty and is a semigroup with a natural multiplication operation. The equation for calculation of a symmetry operator follows from the definition itself:

$$F \circ T(x) | (F(x) = 0) = 0.$$  

As a rule, it is impossible to actually determine the semigroup from the foregoing. It is simpler to look for a one-parameter Lie subgroup $T_t$, provided that such exists; we denote by $\alpha$ the infinitesimal operator of the indicated subgroup. The following propositions are obvious:

$$\frac{\partial T_t(x)}{\partial t} = x \circ T_t(x), \quad T_0(x) = x; \quad (1)$$

$$F(x) = 0 \implies ||F(x - t\alpha(x))|| = 0 (|t|). \quad (2)$$

It can be shown that if a certain mapping $\beta: X \rightarrow X$ satisfies condition (2) and determines a one-parameter group according to (1), then that group is a symmetry group. On the other hand, not every operator satisfying conditions (2) determines a symmetry group. What is essential, however, is that the latter fact does not prohibit the use of such operators for the investigation of the equation; indeed, even when a symmetry group is known, its infinitesimal operator is actually used. In light of these considerations we concern ourselves hereinafter with the algebraic properties of the equation.

§1. The Admissible Operator

We consider every operator involved below to be infinitely differentiable in its domain of definition.

**Definition 2.** $\sigma: X \rightarrow X$ is said to be an admissible operator for $F$ if

$$F(x) = 0 \Rightarrow \| F(x + ts(x)) \| = o(|t|).$$

**Theorem 1.** An operator $\sigma$ is admissible if and only if

$$F'(x) \sigma(x) \mid (F(x) = 0) = 0.$$  \hspace{1cm} (3)

We call this expression the governing equation for the admissible operator. The following proposition is pivotal to the ensuing discussion:

**Theorem 2.** If $\sigma$ is an admissible operator, then for any element $a \in K$ it is possible to construct a one-parameter family $a_\sigma(t) \in K$ in such a way that $\| a_\sigma(t) - a \cdot ts(a) \| = o(|t|).

The truth of the latter follows from a lemma, which we now state, on the local mapping of a regular manifold onto a manifold tangent to it [6]. Let $c \in K$, $\Gamma_c = \{c + h | F'(c) h = 0 \}$.

**Lemma.** It is possible to designate a neighborhood $U$ of a point $c$ and mappings $\varphi: U \cap \Gamma_c \rightarrow K$ and $\psi: U \cap K \rightarrow \Gamma_c$ such that $\| \varphi(c \cdot h) - c - h \| = o(\|h\|)$ and $\| \psi(a) - a \| = o(\|a - c\|)$.

Theorem 2 follows from this assertion, because for any value of the parameter $t$ and any $a \in K$ we have $a + to(a) \in \Gamma_a$. Thus, an admissible operator can be used to construct for every solution of the investigated equation a one-parameter family of solutions differentiable with respect to the parameter at the point $t = 0$. Ultimately it is this latter fact that allows us to use admissible operators for calculation of the particular solutions and qualitative analysis of the given equation.

**Theorem 3.** For any integer $n \geq 0$ and any $a \in K$ the following estimates hold:

$$\| F \circ z_t(a) \| = o(|t|^p); \quad z_t(a) = \sum_{0}^{n} (rt)^{-1} t^d_f(x), \quad d_f(x) = f'(x) \circ (r).$$

**Proof.** We first write the obvious equality

$$\| F \circ z_t(x) - \sum_{0}^{n} (rt)^{-1} t^d_f(F(x)) \| = o(|t|^p).$$

From this expression we infer that the theorem is equivalent to the proposition

$$a \in K \Rightarrow d^a_f(X) | (x = a) = 0.$$ \hspace{1cm} (4)

We prove the latter by induction; since condition (4) is satisfied for $r = 0$, we assume it to be true for $r = m$. Thus, for any values of the parameter we have

$$a \in K \Rightarrow d^a_m(F(x) | (x = a_t(t)) = 0.$$ \hspace{1cm} (4)

The left-hand side of the identity is differentiable with respect to the parameter at the point $t = 0$, so the function $a_\sigma(t)$ also has this property; therefore,

$$a \in K \Rightarrow (d^a_m(F(x)) \circ (x = a) = 0.$$ \hspace{1cm} (4)

Consequently, condition (4) is valid for $r = m + 1; \; this \; proves \; the \; theorem.$ As a rule, for specific calculations it is difficult to use the governing equation in the form (3); the condition is formulated on a manifold, rather than over the entire space. This dilemma can be resolved once the following proposition has been proved. Let $Z(z)$ be a Banach space, and let the mapping $\varphi: X \rightarrow Z$ satisfy the condition $\varphi(K) = 0$.}

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