NEW EXACT SOLUTIONS OF THE DIRAC EQUATION

V. G. Bagrov, D. M. Gitman, V. N. Zadorozhnyi, P. M. Lavrov, and V. N. Shapovalov

The search for new exact solutions to the Dirac and Klein–Gordon equations initiated in [1] is continued. New solutions are found for axisymmetric fields and one type of nonstationary field of special configuration. The basic notation and system of units of [1] are retained.

1. General Properties of the Solutions of the Dirac Equation for an Electron in a Combination of a Longitudinal Magnetic Field and Crossed Fields

We consider the motion of an electron in an electromagnetic field whose intensities E and H in some Lorentz coordinate system are related by

\[ E = -[nH], \]

where n is a unit constant vector. The magnetic field H is arbitrary. It is readily seen that the field tensor \( F_{\mu\nu} \) and the isotropic vector \( n^\mu = (1, n) \) satisfy the condition

\[ n^\mu F_{\mu\nu} = 0. \]

The condition (2) is an invariant characteristic of a field of the structure (1), and the fulfillment of (2) indicates that there exists a Lorentz system in which a field satisfying (2) has the structure (1). It follows from the results of [2–4] that a field of the structure (1) has the spin integral of the motion

\[ T = (\sigma P) + (nS), \quad S = m\rho_0\sigma - \rho_4P \]

and, therefore, the wave function can satisfy in addition to the Dirac equation the subsidiary condition

\[ T\psi = \zeta\psi, \quad \zeta = \pm 1. \]

It is readily verified that for the fields (1) there exists one further integral of the motion \((nP)\), which commutes with the operator T. Thus, we shall also make \( \psi \) satisfy the condition

\[ (nP)\psi = (n^\mu P_\mu)\psi = \lambda\psi, \]

\( \lambda \) in (4) and (5) being the same. As is shown in [5], the solution of the Dirac equation for fields of the type (1) can be sought in the block form

\[ \psi = N\left(\begin{array}{c} m + \lambda + (\sigma P_\perp)(\sigma n) \\ m - \lambda \end{array}\right) \sum_{\zeta} \left[ 1 + \zeta (\sigma n) \Phi_{\zeta} \right] v_\zeta, \quad P_\perp = P - n(nP), \]

where \( v_0 \) is an arbitrary constant two-component spinor, and the scalar functions \( \Phi_{\zeta} \) are solutions of the second-order equations

\[ [P^\mu P_\mu - m^2 + \zeta (nH)] \Phi_{\zeta} = 0. \]

If we require that (6) satisfy (4), we find that Eq. (4) is equivalent to the following condition on the spinor \( v_0 \):

\[ (\sigma n) v_\zeta = \zeta v_\zeta, \]

and \( \zeta = \pm 1 \) in (6) and (8) is the same eigenvalue as in (4). Thus, (6) is a solution to the Dirac equation that is an eigenfunction for the operator (3) when (7) and (8) are satisfied.

In particular, we note that Redmond's field [6] satisfies the condition (1) and
therefore has the spin integral (3), which was pointed out for the first time in [4]. It is noteworthy that the explicit classification of the functions (6) with respect to the spin states does not require knowledge of the explicit form of the solutions of Eqs. (7).

2. Motion of a Charge in an Electromagnetic Field of Axial Symmetry

We introduce the cylindrical coordinate system

\[ x_0 = u_0, \ x_1 = u_1 \cos u_2, \ x_2 = u_1 \sin u_2, \ x_3 = u_3, \]

and in the coordinate system (9) we specify the curvilinear components of the potentials in the form

\[ A_0 = A_0(u_1), \ A_1 = 0, \ A_2 = A_2(u_1), \ A_3 = A_3(u_1), \]

where \( A_i \) (\( i = 0, 2, 3 \)) are arbitrary functions of the variable \( u_i \).

It is easy to find the projections of the fields corresponding to the potentials (10) and of the currents onto the unit vectors of the cylindrical coordinate system:

\[
4\pi = A_0^* + u_1^{-1}A_0', \ f_r = 0, \ 4\pi j_\phi = (u_1A_2 - A_2')u_1^{-2}, \ 4\pi j_z = - A_3 - u_1^{-1}A_3'.
\]

We consider motion of the charge in the field (11).

The classical equations of Lorentz admit in explicit form three first integrals of the motion \( \kappa_1 \) and can be written in the form of the following system of first-order equations:

\[
mu_0 - P_0 = 0, \ m^2u_1^2 = R = P_0^2 - P_3^2 - u_1^{-2}P_2^2 - m^2, \ m^2u_2u_2 + P_2 = 0, \ m^2u_3 + P_3 = 0, \ P_1 = \kappa_1 + A_1.
\]

It is obvious that the system (12) can be integrated and expressed in quadratures:

\[
u_0 = \int pP_0du_1, \ u_2 = - \int p\bar{u}_1^{-2}P_2du_1, \ u_3 = - \int pP_3du_1, \ v = m\int pdu_1, \ p = R^{-1/2}.
\]

The classically allowed region of motion is determined by the condition \( R \geq 0 \).

A complete integral of the Hamilton–Jacobi equation – the classical action function – can also be found by quadrature:

\[
S = S_0 - \int p^{-1}du_1, \ S_0 = \kappa_0u_0 + \kappa_2u_2 + \kappa_3u_3.
\]

The solution to the Klein–Gordon equation for the fields (11) corresponding to definite values of the integrals of the motion \( \kappa_1 \) can be written in the form

\[
\Psi = N \exp(-iS_0)\varphi(u_1), \ \varphi'' + u_1^{-1}\varphi' + R\varphi = 0,
\]

where \( \kappa_2 = l \), in which \( l \) is an integer.

The solution to the Dirac equation can be conveniently represented in the form

\[
\Psi_D = N \exp(-iS_0)(1 + i\gamma_2)\exp\left(-\frac{i}{2}\gamma_2\gamma_2\right)\varphi(u_1),
\]

where the spinor \( \psi(u_1) \) is a solution of the equation

\[
D\psi = 0, \ D = P_0 + i\gamma_1P_1 - \gamma_2u_1^{-1}P_2 + \gamma_3P_3 - m\gamma_3,
\]

where \( \kappa_3 = l - \frac{1}{2}, \ P_1 = \partial_1 + (2u_1)^{-1} \).

Hitherto, solutions to Eqs. (13) and (15) were known only for the case \( A_0 = 0 \); namely, for constant and homogeneous magnetic field and for two types [7-10] of magnetic fields of special configuration.

We have succeeded in finding solutions for fields of new configurations. Solutions to the Klein–Gordon and Dirac equations have hitherto been found only for the cases when the potentials \( A_0 \) and \( A_2 \) are linearly dependent. Under this condition, it is obvious that, applying a Lorentz transformation, one can reduce the problem to the finding of solutions in fields of the following three simplest types.