ALGEBRAIC CLASSIFICATION OF THE MATTER TENSOR

I. M. Dozmorov

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A study is made of the "matter tensor", which is constructed using only the energy tensor and the metric tensor and has all the algebraic properties of the Riemann tensor. The possible types of matter tensor are classified in the same way as Petrov's classification for the Weyl tensor, and the relationship between the matter tensor and the canonical forms of the corresponding energy tensors is established.

The algebraic classification of Einstein spaces formulated by Petrov [1] is a fundamental contribution to the general theory of relativity. By means of this classification it was possible to establish the relationship between the algebraic properties of the curvature tensor and the physical properties of the corresponding gravitational field; this is especially valuable because of the absence in general relativity of a satisfactory definition of the dynamical field characteristics such as field strengths, energy tensor, etc. The physical meaning of the gravitational fields of different Petrov types is usually elucidated by comparing the canonical forms of the curvature tensor and the energy tensor (as a rule, of the electromagnetic field) [2, 3] in flat space. However, a direct comparison of the curvature tensor of fourth rank and the energy tensor of second rank is not always convincing. It is therefore expedient to compare the curvature tensor with a certain tensor of the same rank and with the same symmetry properties in flat space constructed from the energy tensor.

Let us dwell first in more detail on the classification of the curvature tensor in a Riemannian space with signature (---+); it can be split into two parts:

\[ R_{\mu\nu\sigma\tau} = C_{\mu\nu\sigma\tau} + S_{\mu\nu\sigma\tau}, \]

where \( C_{\mu\nu\lambda\sigma} \) is the Weyl conformal curvature tensor with vanishing trace:

\[ C_{\mu\nu\lambda\sigma} R^{\lambda\sigma} = 0 \]

and symmetry properties

\[ C_{\mu\nu\lambda\sigma} = C_{\lambda\mu\nu\sigma} = - C_{\nu\lambda\mu\sigma} = - C_{\sigma\nu\lambda\mu}. \]

The tensor \( S_{\mu\nu\lambda\sigma} \) has the form

\[ S_{\mu\nu\lambda\sigma} = \frac{1}{2} (g_{\mu\lambda} R_{\nu\sigma} - g_{\mu\sigma} R_{\nu\lambda} + g_{\nu\lambda} R_{\mu\sigma} - g_{\nu\sigma} R_{\mu\lambda}) - \frac{1}{6} R (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}). \]

When \( R_{\mu\nu} = 0 \) the Riemann curvature tensor coincides with the Weyl tensor, so that empty Einstein spaces are classified in accordance with the canonical forms of the corresponding Weyl tensors.

In the six-dimensional bivector space \( E_6 \) the Weyl tensor with allowance for (2) and (3) can be reduced to the form

\[ C_{\lambda\mu} = \begin{vmatrix} M & N \\ N & -M \end{vmatrix}, \]

where \( M \) and \( N \) are \( 3 \times 3 \) matrices, \( \lambda = \mu \) and \( \lambda \) are collective indices. Since the matrix \( Q = M + iN \) can be of only three possible types, empty Einstein spaces can also be of only three types with the characteristics [111], [21], [3] of the matrix, respectively, for types I, II, and III.

Penrose [4] pointed out the possibility of a more detailed classification with allowance for possible coincidences of the eigenvalues (stationary curvatures) of the Weyl tensor, as represented in the diagram

\[ \begin{align*}
\Pi_{[3]} &= -D_{(1,1)} \\
\Pi_{[1]} &= -N_{[1]}(1) = 0
\end{align*} \]

(6)

Other methods of classification leading to the same results were considered in [5-9] (see also [10]). The "hierarchy" of the classification is very clearly manifested in the following relations for the Weyl tensor of different types [3, 11]:

\[ \begin{align*}
N_{[3]} l_1 &= 0, \\
\Pi_{[3]} l_1 &= 0, \\
D_{[3]} l_1, l_1 l^* l^* &= 0 \quad (\text{two solutions}) \\
\Pi_{[2]} l_1, l_1 l^* l^* &= 0, \\
l_1, l_1 l_2, l_1 l_3, l_1 l_3 l^* l^* &= 0.
\end{align*} \]

(7)

Here the square brackets denote alternation and the symbols N, III... denote the Weyl tensors of the corresponding types. The isotropic vector \( l^\alpha \) is called the Debever vector or the principal vector of the Weyl tensor. A Weyl tensor that satisfies one of these conditions satisfies all the following ones. The first of the conditions (7) that it satisfies determines its algebraic type. The first four conditions determine "algebraically special fields"; type I is said to be "algebraically general."

The relations (7) enable one to formulate a "composition theorem" [3]. The sum of two or more Weyl tensors for fixed \( g_{\mu\nu} \) and \( l_\mu \) is as special as the most general (in the sense of (7)) of the terms. For example, a sum of III, D, and II tensors belongs to the second Petrov type.

Different authors have also considered the classification of nonempty Riemannian spaces \((R_{\mu\nu} \neq 0)\) [12, 13]. With allowance for Einstein's equations

\[ R_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}. \]

(8)

the tensor \( S_{\mu\nu\lambda\sigma} \) in (4) can be expressed solely in terms of the energy tensor of the matter and the metric tensor. It is assumed [14] that the two terms in (1) represent, respectively, the free gravitational field and its material sources. The relationship between them follows from the Bianchi identities. Locally they are completely independent. Therefore, in a classification of gravitational fields with sources that are local in nature one must consider the algebraic types of the tensor \( S_{\mu\nu\lambda\sigma} \) as well and, since it, like the Weyl tensor, can be of only three different types [13], gravitational fields with sources may have \( 3 \times 3 = 9 \) different types.

Let us consider the properties of the tensor \( S_{\mu\nu\lambda\sigma} \) when \( R_{\mu\nu} \) and \( R \) are replaced by \( T_{\mu\nu} \) and \( T \) from Eqs. (8). Thus, we have at our disposal a matter tensor that depends solely on the energy tensor and has all the algebraic properties of the Weyl tensor except (2). Investigation of the algebraic types of \( S_{\mu\nu\lambda\sigma} \) and their relationship to the corresponding types of the energy tensor enables us, as we hope, to represent better the physical meaning of the Petrov classification of gravitational fields. It is more convenient to perform the analysis using a quasiothogonal Newman-Penrose tetrad [15] (see also [16]) \( Z_{\mu\nu} = (l_\mu, n_\mu, m_\mu, \bar{m}_\mu) \), where

\[ l_\alpha n^\alpha = m_\alpha m^\alpha = 1. \]

(9)

and the remaining contractions are zero. The bar denotes the complex conjugate. The relationship to the metric has the form \( g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu m_\nu - \bar{m}_\mu \bar{m}_\nu \).

Following [15], we introduce notation for the tetrad projections of the Ricci tensor:

\[ \Phi_{\alpha\beta} = -\frac{1}{2} R_{\alpha\beta}, \quad \Phi_{\alpha l} = -\frac{1}{2} R_{\alpha l}, \quad \Phi_{\alpha m} = -\frac{1}{2} R_{\alpha m}, \]

\[ \Phi_{11} = -\frac{1}{4} (R_{12} + R_{13}), \quad \lambda = \frac{R}{24} = \frac{1}{12} (R_{12} - R_{13}), \]

\[ \Phi_{12} = -\frac{1}{2} R_{12}, \quad \Phi_{22} = -\frac{1}{2} R_{22}, \quad \Phi_{\mu\sigma} = \Phi_{\mu\sigma}. \]

(10)