The authors investigate the forced oscillations of a torsional pendulum oriented in a definite way on a platform traveling in a "circular" nonequatorial orbit around a source of a Lenze-Thirring metric (in particular, around the earth). This general relativistic effect is associated with the inception in the platform reference frame of periodic inertial forces acting on the ends of the pendulum, with a period equal to the time of rotation of the platform in its own reference frame.

1. In a Lenze-Thirring metric, using the Hamilton-Jacobi method, we find the trajectory (geodesic) of the center of mass of a platform moving in an orbit of radius \( r = R = \text{const} \) and forming an angle \( \theta_0 \) with the direction of the angular momentum of the field source. The action function \( S \), as usual, has the property \( \partial S / \partial x^\mu = -u_\mu \), where \( u_\mu \) denotes the covariant 4-velocity components of the platform, i.e.,

\[
\frac{dS}{dx^\mu} = -u_\mu dx^\mu = -u_0 dx^0 - u_1 dx^1 - u_2 dx^2 - u_3 dx^3.
\]

We denote \( u_0 = \varepsilon, u_1 = -f, u_2 = -\Theta, u_3 = -\lambda \), where \( \varepsilon \) and \( \lambda \) have the significance of the energy and the projection of the orbital angular momentum onto the direction of the angular momentum of the source, divided by \( m \) and \( c \) in the appropriate power. Then

\[
\begin{align*}
\varepsilon &= u_0 g^{00} = \varepsilon - r g \alpha, r^2, u_1 = -f, \\
\lambda &= u_2 r^2, u_3 = \lambda/r^2 \sin^2 \theta + r g \alpha, r^2.
\end{align*}
\]

where \( \gamma = 1 - r g / r \). The Hamilton-Jacobi equation \( g^{\mu\nu}(\partial S / \partial x^\mu)(\partial S / \partial x^\nu) = 1 \) corresponds to the relation \( g^{\mu\nu} u_\mu u_\nu = 1 \) and implies

\[
\varepsilon^2 / \gamma - f^2 / \gamma - \Theta^2 / r^2 - \lambda^2 / r^2 \sin^2 \theta - 2 r g \alpha / r^2 = 1.
\]

(2)

Knowing that \( x^0 \) and \( \varphi \) are cyclic coordinates in (2), we write the action function in the form \( S = -\varepsilon x^0 + \int f(r) dr + \int \Theta (\theta) d\theta + \lambda \varphi \), whereupon the variables are separable in Eq. (2):

\[
\begin{align*}
\rho^2 [\varepsilon^2 / \gamma - f^2 / \gamma - 2 \rho g \alpha / r^2 - 1] &= \varepsilon^2 / \gamma + \lambda^2 / r^2 \sin^2 \theta = \mu^2, \\
\end{align*}
\]

where \( \mu = \text{const} \) is the separation parameter and has the sense of the orbital angular momentum; \( \mu = \lambda / \sin \theta \). The quantities \( \varepsilon \) and \( \lambda \) entering into \( U^\mu \) (first integrals of the Hamilton-Jacobi equation) can be found by Synge's procedure [1]. Neglecting terms of order \( \alpha \rho^2 / R \) and higher, we obtain as a result

\[
\begin{align*}
\mu &= \pm 2 R^2 \left( 1 - 3 \rho g 4 R - 3 \alpha \Omega \sin \theta \right) \sin \theta, \\
\varepsilon &= 1 - r g 4 R, \quad \Omega = \pm \frac{1}{r g} \sqrt{2 R^2},
\end{align*}
\]

where the plus sign is taken for motion in the positive direction of the orbit (i.e., with increasing \( \varphi \)), and the minus sign for motion in the opposite direction. The first integrals (1) can be used to deduce the relations

\[
\Omega = \frac{d \varphi}{dx^0} = \frac{\mu^2}{\mu^3} = \frac{\sin \theta}{\sin \theta + \rho g / k^2},
\]

(3)

\[
\Omega_\theta = \frac{d\theta}{dx^0} = \frac{\mu^2}{\mu^0} = \sqrt{\frac{1}{\sin^2 \theta}} \left( 1 - \frac{\sin^2 \theta}{\sin^2 \theta} \right),
\]

in which \( \omega = \Omega(1 - 3\Omega \sin \theta_0) \); here the plus sign refer to "downward" motion, i.e., when \( \theta \) is increasing, and the minus sign to "upward" motion. Integrating (3) and (4), we obtain the second integrals of the Hamilton-Jacobi equation

\[
\varphi = \arcsin \left( \frac{\cot \theta}{\cot \theta_0} \right) + \frac{r_0^2}{R^2} x^0,
\]

\[
\cos \theta = \cos \theta_0 \cos \omega x^0.
\]

The second terms on the right-hand sides of (3) and (5) correspond to Lenze-Thirring precession, i.e., precession of the orbital angular momentum vector relative to the angular momentum of the central body.

2. For transition to a reference frame associated with the rotating platform we make three successive coordinate transformations.

a) We first make a transformation tracking the Lenze-Thirring precession, i.e., such that the orbit plane is at rest in the primed coordinate system:

\[
x' = x, \quad r' = r, \quad \theta' = \theta, \quad \varphi' = \varphi - \frac{r_0^2}{R^2} x^0,
\]

whereupon

\[
\Omega_\varphi' = \frac{\omega \sin \theta}{\sin^2 \theta}, \quad \sin \varphi' = \frac{\sin \varphi}{\tan \theta_0} \cot \theta,
\]

\[
\Omega_\theta' = \Omega_\theta.
\]

b) Next we rotate the coordinate system through an angle \( \pi/2 - \theta_0 \), aligning the equatorial plane with the orbit plane:

\[
x'' = x', \quad r'' = r', \quad \cos \theta'' = \cos \theta' \sin \theta_0 - \sin \theta' \sin \varphi' \cos \theta_0
\]

\[
\tan \varphi'' = \tan \varphi' \sin \theta_0 + \cos \theta' \sin \theta_0 \sin \theta' \cos \varphi'.
\]

In this coordinate system \( \Omega_\varphi'' = \omega, \quad \Omega_\theta'' = 0. \)

c) Finally, we transform to a coordinate system rotating together with the platform, where the center of mass has constant coordinates:

\[
x''' = x'', \quad r''' = r'', \quad \theta''' = \theta'', \quad \varphi''' = \varphi'' - \Omega_\varphi x''.
\]

3. We associate with the final coordinate system a chronometric [2] reference frame. It is specified by a set of observers in the vicinity of the platform, whose 4-velocity components in the given coordinate system have the form \( u_\mu'' = v_0''/\sqrt{\gamma_0''}, \quad u'' = v_0''/\sqrt{\gamma_0''} \). We drop the primes from now on. We assume that the torsional pendulum represents a rod with its entire mass concentrated at the ends. The chronometrically invariant equation for the world lines of the masses has the form [3]

\[
-d\rho^i/d\tau = L_{\alpha k} p^\alpha p^k + m F^i + 2p^k (A^j_k - D^j_k) + \Phi^i,
\]

where \( d\tau = \sqrt{\gamma_0} dx^0 \) is a time interval in the chronometric frame; \( v_k = dx_k/d\tau \) and \( p^i = mv^i \) are the three-dimensional velocity and momentum, \( m = m_0/\sqrt{1 - v^2} \); \( F^i, A^j_k, \) and \( D^j_k \) are the acceleration, angular velocity tensor of rotation, and tensor of deformation rates of the frame; \( L_{\alpha k} \) denotes the chronometrically invariant three-dimensional connectivity coefficients; and \( \Phi^i \) is the nongravitational force.

4. We investigate the oscillations of the pendulum in two characteristic orientations, assuming that the deviations from the equilibrium position and the 3-velocities of its ends are small, with the same order of smallness.

a) In its equilibrium position let the pendulum be oriented along the radius vector of the center of mass of the platform (r orientation), and let its axis be fixed in such a way...