RADIATION OF RELATIVISTIC FERMIONS IN A PERIODIC MAGNETIC FIELD

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The study of the quantum nature of the motion of charged particles in periodic magnetic field has, as is well known, great theoretical and practical importance. The present wide use of undulators opens new possibilities for experimental investigation of quantum effects. The quantum mechanical nature of the motion of charged particles in a periodic magnetic field has been investigated in several theoretical works in which both specific types of magnetic field [1] and some general properties of undulator radiation in the quasiclassical approximation have been considered, with consideration of the summation over electron spin [2, 3].

The wave function of an electron in a periodic field, separated by polarization state, in the near-axis approximation is found in this work with the help of the method of [4]. The use of the explicit expression for the wave function in investigating the electron radiation permitted obtaining more complete results in the comparison with other approximate methods.

1. WAVE FUNCTION

We will describe the motion of an electron in a periodic magnetic field on the basis of the Dirac equation. The vector potential of the magnetic field is written in Cartesian coordinates in the form

\[ \vec{A} = (0, A(x), 0); \quad A(x + 2\pi) = A(x), \]

where \( A(x) \) is an arbitrary continuous periodic function with period \( 2\pi \).

The solution of the Dirac equation

\[ \left( i\hbar \frac{\partial}{\partial t} - \mathbf{e}(\mathbf{P}) - \gamma \gamma_5 \right) \psi(r, t) = 0, \]

where \( \mathbf{P} = -i\hbar \mathbf{\nabla} + \frac{e_0 A}{\hbar c} \) is the momentum operator, \( \gamma \) and \( \gamma_5 \) are the Dirac matrices, \( e = -e_0 < 0 \) and \( m_0 \) are the charge and mass of the electron, will be found in a form which permits separating the spin coefficients (see also [5]):

\[ \psi(r, t) = e^{-i/k_{\perp} r_{\perp} + i k_{\perp} z} \begin{pmatrix} C_0 f(x) \\ C_0 g(x) \\ iC_1 f(x) \\ iC_1 g(x) \end{pmatrix} \]

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The functions \( f(x) \) and \( g(x) \) then satisfy the system of equations:

\[
\begin{aligned}
\frac{d}{dx} + \chi(\kappa, x) \mu &= i\mu, \quad \chi(\kappa_2, x) = \kappa_2 + \frac{e_0 A(x)}{\hbar} ; \\
\frac{d}{dx} - \gamma(\kappa_2, x) f &= -i g.
\end{aligned}
\]  
(4)

The spin coefficients \( C_j \) \((j = 1, 2, 3, 4)\) satisfy the system of algebraic equations:

\[
\begin{aligned}
((K - i\kappa_0) C_{1,2} - i C_{1,2} - \kappa_1 C_{2,2} = 0; \\
((K - i\kappa_0) C_{1,4} - i C_{1,4} + \kappa_3 C_{3,4} = 0.
\end{aligned}
\]  
(5)

System (5) has a determinant equaling zero under the condition

\[
K^2 = \kappa^2 + \kappa^2_0 + \kappa^2_1,
\]  
(6)

where \( E = \hbar K \) is the electron energy and \( k_0 = m_e c/\hbar \).

In order to find the electron wave functions separated by polarization states with respect to the direction of the magnetic field (transverse polarization), we impose an additional condition on the wave function (3):

\[
\psi = m_e c^2 + \gamma_2 \sigma, \quad P, \quad K_0 = \sqrt{K^2 - \kappa^2_1},
\]  
(7)

The combined solution of systems (5) and (7) with normalization taken into account leads to the following expression for the spin coefficients \( C_j \) (see [6]):

\[
\begin{pmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix}
(A_1 + A_2) B_2 \\
(A_1 - A_2) B_1 \\
(A_2 - A_1) B_3 \\
(A_2 + A_1) B_4
\end{pmatrix},
\]  
(8)

where

\[
A_2 = \sqrt{1 + \frac{\kappa_3}{K}}, \quad A_1 = \sqrt{1 - \frac{\kappa_3}{K}}; \\
B_2 = \sqrt{1 + \frac{\kappa_0}{K}}, \quad B_1 = \sqrt{1 - \frac{\kappa_0}{K}}.
\]

The quantity \( \zeta = \pm 1 \) characterized the orientation of the spin along the positive \((\zeta = 1)\) and negative \((\zeta = -1)\) OZ axis.

To find the solution of system (4), we will assume that the classical trajectory of the electron passes near the axis of the undulator, i.e., the near-axis approximation, which is important from the point of view of practical application of undulators, is realized. In this case

\[
|M \cdot a^{-1}| \ll 1; \quad M = \max \left\{ \kappa_2, \frac{e_0 A(x)}{\hbar} \right\}; \quad x \in [-a, a],
\]  
(9)

where \( p_1 = \hbar k^1 \) is the average momentum of the electron along the OX axis.

The solution of system (4) will be found in the form of an expansion in the small parameter \( \alpha = 1/k_1 \), which will mean the actual expansion in the quantity (9).

We will represent system (4) in matrix form with an accuracy to terms containing \( \alpha^3 \):