WEAK CYLINDRICAL GRAVITATIONAL WAVES

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A study is made of the linearized Einstein equations without right-hand side on the background of a flat metric in cylindrical coordinates. General solutions are obtained by an expansion with respect to irreducible representations of the group of cylindrical symmetry. The solutions are investigated with allowance for additional conditions imposed on the physico-geometrical quantities, a monad choice being made of the frame of reference in the chronometric and kinemetric gauges.

INTRODUCTION

There is a large class of problems, for example, astronomical problems within the solar system, in which approximate solutions of Einstein's equations are adequate. However, without additional conditions even an approximate problem is not completely determined. Symmetry properties of the solutions are frequently taken for the additional conditions.

Symmetry properties are particularly important in the description of the quantized gravitational field. In this case, symmetry amounts to observability of definite field characteristics. For example, expansion in plane waves corresponds to a description of the field by means of the momentum and spin projection; spherical symmetry, to description by the orbital angular momentum and the spin [1]; and cylindrical symmetry, to description by the orbital angular momentum, the projection of the momentum, and the spin.

The use of a definite symmetry to find approximate solutions of Einstein's equations enables one to separate the variables and frequently to reduce the differential equations to algebraic equations if one seeks solutions in the form of expansions with respect to functions that realize irreducible representations of the group of the given symmetry [2].

In this paper, we investigate solutions with cylindrical symmetry, concentrating our main attention on nonstationary metrics. A number of exact "wave" solutions of Einstein's equations with axial symmetry is known [3, 4]. Nonstationary axisymmetric fields were investigated by Weber and Wheeler [5].

It is well known that a solution of Einstein's equation is determined to within four arbitrary functions. This freedom is interpreted as the possibility of choosing an arbitrary frame of reference. In this paper, we investigate additional conditions associated with the chronometric [6-8] and kinemetric [9-11] monad gauge.

1. Einstein's Equations for Weak Cylindrical

Gravitational Waves

We shall consider the linearized Einstein equations without right-hand side with background flat metric in cylindrical coordinates $x^0, r, \varphi, z$. The background metric is diagonal:

$$\gamma_{\mu\nu} = \text{diag}(1, -1, -r^2, -1).$$

We seek solutions in a form that realizes an expansion of the functions with respect to irreducible representations of the group of motions of the Euclidean plane [2]:

$$a_{\mu}(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} a_{\mu}(\omega, \kappa, n, \lambda)(-i)^n j_n(\lambda r),$$

where $j_n$ is the Bessel function of the first kind.

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where $J_n(\lambda r)$ is a Bessel function with integral index.

We present the metric in the form $g_{\mu\nu} = \varepsilon_{\mu\nu} + y_{\mu\nu} + O(y)$. To separate the variables in the Einstein equations $R_{\mu\nu} = 0$, we must represent these equations and the metric $y_{\mu\nu}$ in the form of combinations that are irreducible with respect to the group of motions of the Euclidean plane $(r, \varphi)$. Irreducible combinations of the metric are

\begin{align*}
    u_{06} &= y_{06}, \quad u_{03} = e^{i\psi} \left( y_{06} + \frac{i}{r} y_{22} \right), \quad u_{01} = u_{03}^*, \quad u_{05} = y_{05}, \\
    u_{11} &= e^{i\psi} \left( y_{11} + \frac{2i}{r} y_{12} - \frac{1}{r^2} y_{22} \right), \quad u_{12} = u_{11}^*, \quad u_{13} = y_{11} + \frac{1}{r} y_{22}, \\
    u_{15} &= e^{i\psi} \left( y_{15} + \frac{i}{r} y_{23} \right), \quad u_{23} = u_{15}^*, \quad u_{33} = y_{33},
\end{align*}

where the asterisk denotes the complex conjugate. We introduce the operators $H_+$ and $H_-$, which couple the irreducible representations of the group of motions of the Euclidean plane: $H_+ = -e^{i\psi}(\partial/\partial r + (i/r)(\partial/\partial \varphi))$, $H_- = H_+^*$. These operators have the properties

\begin{align*}
    H_+ H_- - H_- H_+ &= 0, \\
    H_+ H_- &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}, \\
    H_+^2 &= e^{2i\psi} \left[ \frac{\partial^2}{\partial r^2} + \frac{2i}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} - \frac{1}{r^2} \frac{\partial}{\partial r} - \frac{2i}{r} \frac{\partial}{\partial \varphi} \right], \\
    H_+ H_- \Phi^+_m &= -\lambda^2 \Phi^+_m, \\
    H_+ \Phi^+_m = \pm i \lambda \Phi^+_m, \quad H_- \Phi^+_m = \pm i \lambda \Phi^+_m, \\
    \Phi^+_m &= (-i)^m e^{i\psi} \Phi_m(\lambda r), \quad \Phi^-_m = (\Phi^+_m)^*.
\end{align*}

The irreducible combinations of the Einstein equations are

\begin{align*}
    R_{00} &= \frac{1}{2} \left( H_+ H_- + \frac{\partial^2}{\partial x^2} \right) u_{00} + \frac{1}{2} \frac{\partial}{\partial x^3} H_- u_{01} + \frac{1}{2} \frac{\partial}{\partial x^0} H_+ u_{03} + \frac{1}{2} \frac{\partial}{\partial x^0} H_- u_{03}, \\
    \frac{\partial^2}{\partial x^0 \partial x_2} u_{03} + \frac{1}{2} \frac{\partial}{\partial x^0} u_{12} + \frac{1}{2} \frac{\partial}{\partial x^0} u_{23} &= 0, \\
    2 \left( R_{01} + \frac{i}{r} R_{03} \right) e^{i\psi} &= \left( \frac{\partial^2}{\partial x^2} + \frac{1}{2} H_+ H_- \right) u_{01} - \frac{1}{2} H_+^2 u_{03} + \frac{\partial}{\partial z} H_+ u_{03}, \\
    + \frac{1}{2} \frac{\partial}{\partial x^0} H_- u_{11} - \frac{1}{2} \frac{\partial}{\partial x^0} H_+ u_{12} - \frac{\partial^2}{\partial x^0 \partial z} u_{13} - \frac{\partial}{\partial x^0} H_+ u_{23} &= 0, \\
    2 R_{03} &= \frac{1}{2} \frac{\partial}{\partial z} H_- u_{01} + \frac{1}{2} \frac{\partial}{\partial z} H_+ u_{03} + \frac{\partial^2}{\partial x^0 \partial z} u_{13} + \frac{1}{2} \frac{\partial}{\partial x^0} H_+ u_{23} = 0, \quad (3)
\end{align*}

\begin{align*}
    2 \left( R_{13} + \frac{i}{r} R_{23} \right) e^{i\psi} &= \frac{\partial}{\partial z} H_+ u_{00} + \frac{\partial^2}{\partial x^0 \partial z} u_{01} - \frac{\partial}{\partial x^0} H_+ u_{03}, \\
    + \frac{1}{2} \frac{\partial}{\partial z} H_- u_{11} - \frac{1}{2} \frac{\partial}{\partial z} H_+ u_{12} + \left( \frac{\partial^2}{\partial x^0 \partial z} - \frac{1}{2} H_+ H_- \right) u_{13} - \frac{1}{2} H_+^2 u_{23} &= 0, \quad (6)
\end{align*}

\begin{align*}
    2 \left( R_{11} + \frac{1}{r^2} R_{22} \right) &= -H_+ H_- u_{00} - \frac{\partial}{\partial x^0} H_- u_{01} - \frac{\partial}{\partial x^0} H_+ u_{02} - \frac{1}{2} H_-^2 u_{11}, \\
    - \left( \frac{\partial^2}{\partial x^0 \partial z} - \frac{\partial^2}{\partial x^2} - H_+ H_- \right) u_{12} + \frac{\partial}{\partial z} H_- u_{13} - \frac{1}{2} H_+^2 u_{23} + \frac{\partial}{\partial z} H_+ u_{23} + H_+ H_- u_{23} &= 0, \\
    \left( -R_{11} + \frac{1}{r} R_{22} - \frac{2i}{r} R_{12} \right) e^{i\psi} &= \frac{1}{2} H_+^2 u_{00} + \frac{\partial}{\partial x^0} H_+ u_{01} + \frac{1}{2} \frac{\partial}{\partial x^0} \frac{\partial^2}{\partial z^2} u_{11} - \frac{\partial}{\partial z} H_+ u_{13} - \frac{1}{2} H_+^2 u_{23} = 0, \\
    R_{23} &= -\frac{1}{2} \frac{\partial^2}{\partial x^0 \partial z} u_{00} + \frac{\partial}{\partial x^0} \frac{\partial^2}{\partial z^2} u_{01} + \frac{1}{2} \frac{\partial}{\partial x^0} u_{13} + \frac{1}{2} \frac{\partial}{\partial z} H_- u_{13} \\
    + \frac{1}{2} \frac{\partial}{\partial z} H_+ u_{23} - \frac{1}{2} \left( \frac{\partial^2}{\partial x^0 \partial z} - H_+ H_- \right) u_{23} &= 0. \quad (9)
\end{align*}