THEORY OF ELASTIC WAVES IN REAL CRYSTALS. I

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A new approach to the description of wave processes in real crystals is described; in this approach, the equations of motion of a continuous medium are written as a system of first-order partial differential equations in terms of the displacement vector. Some consequences which do not follow directly from the initial equations of motion are analyzed. It is shown, in particular, that the boundary conditions for the displacement field may be obtained directly from the processed equations. The analogy with electromagnetic waves is briefly discussed.

In the absence of long-range forces, the equations of motion of a continuous medium may be written, in a linear approximation, in the form [1, 2]*

\[ \rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ik}}{\partial x_k}, \]  

(1)

where \( \rho \) is the density of the material; \( u_i \) are the components of the elastic-displacement vector; \( \sigma_{ik} \) are the components of the stress tensor.

If the strain is thermodynamically reversible, Eq. (1) forms a complete system with Hooke's law

\[ \sigma_{ik} = c_{iklm} \varepsilon_{lm}, \]

(2)

where \( c_{iklm} \) is the elastic-modulus tensor; \( \varepsilon_{lm} \) are the components of the strain tensor

\[ \varepsilon_{lm} = \frac{1}{2} \left( \frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l} \right). \]

(3)

It follows from Eqs. (1)-(3) that the elastic-displacement vector satisfies an equation of the form

\[ \rho \frac{\partial^2 u_i}{\partial t^2} = c_{iklm} \frac{\partial^2 u_m}{\partial x_k \partial x_l}. \]

(4)

The investigation of wave processes on the basis of Eq. (4) usually reduces to finding the eigenvalues and eigenvectors of the characteristic Christoffel equation obtained from Eq. (4) by transforming to Fourier components. It is then found that for each fixed direction of the wave-normal vector there are, in general, three types of wave propagating with different velocities [1]. The boundary conditions for the displacement vector cannot be derived directly from Eq. (4) and are usually imposed externally as rigid-coupling conditions. The boundary conditions for the stress-tensor components are derived from the equilibrium equation [1, 2].

It was shown earlier [3, 4] that for an isotropic medium Eq. (4) may be written as a system of first-order partial differential equations in terms of the displacement vector, from which the boundary conditions for the stress-vector components may be directly derived. If the crystal lattice is free of defects, the rigid-coupling conditions are identical with the continuity condition for the continuous medium. The boundary conditions for the stress-tensor components may be obtained from the general equations of motion, i.e., from Eq. (1).

The present work generalizes the results of [3, 4] to the case of an arbitrary anisotropic medium. It is shown that Eq. (4) may be formulated as an adequate system of first-order partial differential equations in terms of the displacement vector. On the basis of the resulting equations, some consequences that do not follow directly from Eq. (4) are analyzed.

*In Eq. (1) and everywhere below, summation from 1 to 3 is taken over double Latin subscripts.
First of all, wave processes in ideally elastic media will be considered. Let \( \mathbf{n} \) be the unit vector of the wave normal, the direction of which is fixed arbitrarily in the crystal. By the projective coordinate transformation \( x_i = n_i x_k x_k \), it is simple to establish that for plane waves

\[
\frac{\partial^2}{\partial x_i \partial x_j} = n_i n_j \Delta,
\]

where \( \Delta = \frac{\partial^2}{\partial x_i^2} \) is a Laplace operator.

After making the substitution in Eq. (5), Eq. (4) takes the form

\[
\rho \frac{\partial^2 \mathbf{u}_i}{\partial t^2} = \epsilon_{i j k l} n_j \Delta \mathbf{u}_l.
\]

Following [1], a tensor of reduced elastic moduli \( C_{i m}(\mathbf{n}) \) is now introduced; the components of this tensor depend quadratically on the direction of the wave-normal vector

\[
C_{i m}(\mathbf{n}) \equiv \frac{1}{\gamma} \epsilon_{i k l} n_k \Delta \mathbf{u}_l.
\]

Using Eq. (7), Eq. (6) can be rewritten in the form

\[
\frac{\partial^2 \mathbf{u}_i}{\partial t^2} = C_{i m}(\mathbf{n}) \Delta \mathbf{u}_m.
\]

It follows directly from Eq. (8), after transforming to Fourier components, that the eigenvalues of the tensor \( C_{i m}(\mathbf{n}) \) are the square of the phase velocities \( c_1^2, c_2^2, c_3^2 \) and the eigenvectors are the displacement vectors \( \mathbf{u}_1^{(l)}, \mathbf{u}_2^{(l)}, \mathbf{u}_3^{(l)} \) of three plane isonormal waves. Since the tensor components \( C_{i m}(\mathbf{n}) \) depend explicitly on the components of the wave-normal vector, \( c_2^2 \) and \( \mathbf{u}^{(n)}(\mathbf{n}) \), \( n = 1, 2, 3 \), will also depend on \( \mathbf{n} \). Finding the functions \( c_2^2(\mathbf{n}) \) and \( \mathbf{u}^{(n)}(\mathbf{n}) \), in the general case of an arbitrary anisotropic medium is an intrinsically interesting problem, but is of secondary importance in the present approach. Note that an accurate solution of this problem is only known for an isotropic medium and a hexagonal crystal [1].

From what has been said it follows that in the system of major axes the tensor \( C_{i m}(\mathbf{n}) \) is diagonal and therefore may be written in the form

\[
C_{i m}(\mathbf{n}) = \sum_{n=1}^{3} c_n^2(\mathbf{n}) e^{(n)}_i e^{(n)}_m.
\]

where \( e^{(n)}, n = 1, 2, 3, \) is a triad of orthonormal eigenvalues of the tensor \( C_{i m}(\mathbf{n}) \) defining the direction of the major axes for the given direction of the wave-normal vector.

Substituting Eq. (9) into Eq. (8) gives

\[
\frac{\partial^2 \mathbf{u}_i}{\partial t^2} = \sum_{n=1}^{3} c_n^2(\mathbf{n}) e^{(n)}_i e^{(n)}_m \Delta \mathbf{u}_m.
\]

Taking into account that the total displacement vector \( \mathbf{u} \) is

\[
\mathbf{u} = \sum_{n=1}^{3} \mathbf{u}^{(n)} = \sum_{n=1}^{3} \mathbf{u}^{(n)} e^{(n)}
\]

and using the relation \( e^{(n)}_m e^{(k)}_m = \delta_{nk} \), simple transformations yield the result

\[
\sum_{n=1}^{3} \left[ \frac{\partial^2 \mathbf{u}^{(n)}}{\partial t^2} - c_n^2(\mathbf{n}) \Delta \mathbf{u}^{(n)} \right] = 0.
\]

Since the vectors \( \mathbf{u}^{(n)} \) are linearly independent, Eq. (12) may be resolved into three independent equations.

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* A coordinate transformation is said to be projective if the transformation matrix \( \hat{\mathbf{a}} \) satisfies the relation \( \hat{\mathbf{a}} \cdot \hat{\mathbf{a}} = \hat{\mathbf{a}} \) [5].