
CANONICAL FORMULATION OF THE THEORY OF GRAVITATION
AND A GENERAL-COVARIANT SEPARATION OF
SPACE AND TIME

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A general-covariant separation of space and time is used to divide four-dimensional space into a family of spacelike hypersurfaces. The operation of differentiation with respect to the parameter $t$, which numbers the hypersurfaces, is introduced as a Lie derivative along the congruence of the timelike nonunit vector $\xi$. This method is applied to the variational principle for the gravitational field and is used to construct the corresponding canonical formalism.

Field theory is often recorded in the language of canonical formalism for the purpose of a more convenient transition to quantum theory; this applies to the theory of the gravitational field also [1, 2]. However, in the purely classical, general theory of relativity the canonical formalism is used for an analysis of the Cauchy problem [3]; moreover, one can hope to obtain nontrivial results by determining the canonical transformations for the fields [4], especially the gravitational field, when the development of methods for finding general solutions to the field equations becomes one of the important general problems because of their nonlinearity.

This nonlinearity leads to difficulties also in the canonical approach: when the connection equations are considered, the gravitational field refers to the number of fields with singular Lagrangians [5]. Thus, usual noncovariant approach to the separation of space and time in the canonical formalism [2, 4, 6] can also be considered unsatisfactory. In this paper we consider the latter problem, and we will show how a monad determination of the reference system [7, 8] allows a covariant formulation of the canonical formalism in gravitational

theory. Then the well-known kinematically invariant formulation of the theory [7, 9] follows as a particular case with a special gauge of the monads.

1. Global Separation of Space and Time

We represent the four-dimensional region as a one-parameter family of hypersurfaces \( \Sigma_{t}: t(\chi^\mu) = \text{const} \). Then the unit vector of the normal equals \( \tau_\mu = N \chi^\mu \), where the comma denotes partial differentiation. The tensor \( b_{\alpha\beta} = \varepsilon_{\alpha\beta} - \tau_\alpha \tau_\beta \) as the induced metric on \( \Sigma_{t} \) [3]. A necessary and sufficient condition for the existence of a hypersurface is \( \tau_{[\alpha;\beta]} b_{\mu,\nu} = 0 \), where \( b_{\mu} = \frac{\partial}{\partial x^\mu} - \tau^\mu \) is a projector on three-dimensional space that is orthogonal to \( \tau = \partial/\partial s \). For any geometrical object \( A_B \) (\( B \) is a collective subscript) we define the derivative with respect to time as

\[
L_t A_B = \lim_{\Delta t \to 0} \frac{A_B(p) - A_B(p)}{\Delta t},
\]

where the point \( p \) lies on \( \Sigma_{t} \) for any time \( t \), and \( L_t \) is a Lie derivative along the congruence of the vector \( \xi^\mu = N \chi^\mu = dx^\mu/dt \). The differentiation assumes a comparison between quantities taken at the same "point" in three-dimensional space, but at different instants of "time" (the parameter \( t \)); it is known in the mechanics of continuous media as the determination of the "material rate" of change [10].

We determine the external curvature tensor (the fundamental two-form) on the hypersurface \( \Sigma_{t} \)

\[
\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\alpha\beta}^{(1,2)} = \frac{(1/2N) L_t b_{\alpha\beta}}{L_t b_{\alpha\beta}},
\]

The external curvature describes the embedding of \( \Sigma_{t} \) into ambient space, indicating how the normal changes by its parallel transport along the hypersurface [11]. For the projector \( b_{\alpha\beta} \) we find \( L_t b_{\alpha\beta} = 0 \).

The induced metric \( b_{\mu\nu} \) determines the connectivity on \( \Sigma_{t} \). We denote the covariant derivative with respect to this connection by a vertical bar. For any tensor orthogonal to \( u \) over all its indices we have

\[
\nabla^\mu T_{\mu}^{\alpha\beta\gamma\delta} = 0
\]

It is easy to see that \( b_{\mu\nu}\nabla_{\mu}\nabla_{\nu} = 0 \). Using the above-mentioned differential operations, we separate out the four-dimensional derivatives

\[
A_{\alpha \beta} = A_{\alpha \beta}^{(1,2)} + (\gamma_{\alpha} N) L_t A_{\alpha \beta} - (\gamma_{\beta} N) A_{\alpha \beta}^{(1,2)} = \frac{(1/2N) L_t a_{\alpha \beta}}{L_t a_{\alpha \beta}},
\]

where \( \varepsilon A_{B} T_{\alpha\beta\gamma\delta}^{(1,2)} = A_{B}^{(1,2)}(\chi^{\nu}) - A_{B}(\chi^{\nu}) \) for infinitesimal transformations \( \chi^{\mu} = \chi^{\mu} + \varepsilon \xi^{\mu} \) [12]. We determine the curvature tensor on \( \Sigma_{t} \) according to the following rule [3]:

\[
\Theta_{\alpha \beta} = \Theta^{(1,2)} - \Theta^{(3)} R^{\alpha \beta}_{\gamma \delta},
\]

where \( \Theta \) is an arbitrary vector that is tangential to the hypersurface \( \Sigma_{t} \). The connection between the curvature tensor \( R_{\chi^\alpha \mu \nu} \) and the curvature tensor of the enveloping space is established by means of the Gaussian equations [3]:

\[
R_{\chi^\alpha \mu \nu} = R_{\gamma \delta \mu \nu} + \mathcal{F}_{\chi^\mu \gamma} \mathcal{F}_{\chi^\nu \delta} - \mathcal{F}_{\chi^\nu \gamma} \mathcal{F}_{\chi^\mu \delta}.
\]

Using the connection between the alternative covariant derivatives of the vector \( \tau \) and the curvature tensor \( R_{\gamma \delta} \), we obtain [11]

\[
R_{\tau^\alpha \tau^\mu \tau^\nu \tau^\rho} = L_t \mathcal{F}_{\tau^\nu \tau^\rho} - \mathcal{F}_{\tau^\rho \tau^\nu} - G_{\alpha \tau^\nu} + G_{\nu \tau^\alpha},
\]

where \( G_{\alpha} = \tau_{[\alpha;\gamma]} \). The connection between the fundamental two-form and the curvature tensor of the enveloping space is established by means of the Codazzi equations [3]

\[
\mathcal{F}_{\tau^\alpha \tau^\mu} = - R_{\tau^\alpha \tau^\mu}.
\]

For the scalar curvature we find

\[
R = R - 2R_{\tau^\alpha \tau^\mu \tau^\rho} \tau^\nu \tau^\rho - 2R_{\tau^\alpha \tau^\mu \tau^\nu \tau^\rho} \tau^\nu \tau^\rho + \frac{2}{\sqrt{-g}} L_t (\sqrt{-g} \mathcal{F}) - 2G_{\alpha \tau^\nu} \tau^\nu,
\]

where \( b = -g/N^2 \) is the "determinant" of the induced matrix.