It is shown that the relativistic constraint of the length of possible trajectories in the method of integrals over the trajectories results in the Klein–Gordon equation.

In [1, 2], it was proposed to introduce a weight function $\tau_0 \Delta n_0$ into account the probability of realizing a trajectory in taking account of the contribution of each trajectory in the probability amplitude in the method of integrals over Feynman trajectories [3]. Such trajectory separation results [2] in quasirelativistic corrections to the Schrödinger equation. In this paper the influence of the weight function is examined in a completely relativistic approximation.

The integral equation for the $\psi$-function that is invariant with respect to the Lorentz transformation has the following form for a charged particle with zero spin in an electromagnetic field:

$$\psi (r_2, t_2) = \int_0^t \int K(r_2, t_2; r_1, t_1) \psi (r_1, t_1) d\tau d\tau,$$

where, under the condition of smallness of the parameter $\tau = t_1 - t_0$, the propagator equals

$$K(r_2, t_2; r_1, t_1) = B \exp \left\{ - \frac{i}{\hbar} m_0 c^2 (t_2 - t_1) \sqrt{1 - \frac{(r_2 - r_1)^2}{c^2 (t_2 - t_1)^2}} - \frac{i}{\hbar} \left( \frac{t_2 + t_1}{2} \right) + \frac{i e}{c} \left( \frac{(r_2 - r_1) A \left( \frac{r_2 + r_1}{2}, \frac{t_2 + t_1}{2} \right)}{2} \right) \right\} \times f(r_3 - r_1, t_3 - t_1),$$

$m_0$ is the particle rest mass; $e$ is its charge, $c$ is the velocity of light, $\varphi$ and $A$ are the scalar and vector field potentials, $B$ is a normalizing constant (its value is determined [3] by the rapidly varying part of the propagator), $f(r_2 - r_1, t_2 - t_1)$ is a real function of the interval $s = (c^2 (t_2 - t_1)^2 - (r_2 - r_1)^2)^{1/2}$ that takes account of the weight of each possible trajectory, and it is assumed that $\lim_{s \to 0} f(s) \to 0$ (such an assignment of $f(s)$ automatically takes into account that the trajectory length should be less than $c(t_2 - t_1)$ for a particle with non-zero rest mass).

Using the smallness of $\tau$, we go from the integral to a differential equation. To do this, we expand

$$\exp \left\{ - \frac{i \hbar}{\hbar^2} \left( \frac{t_2 + t_1}{2} \right) + \frac{i e}{c} \left( \frac{(r_2 - r_1) A \left( \frac{r_2 + r_1}{2}, \frac{t_2 + t_1}{2} \right)}{2} \right) \right\}$$

and the wave function under the integral sign in series in $t = t_2 - t_1$ and $r = r_1 - r_2$ to the accuracy of third order infinitesimals and in conformity with the general methodology [3] (after determining the normalizing constant), we extract components independent of $\tau$. Then after integration over the angles we obtain the expression

$$ih \frac{\partial \psi}{\partial t} = e \varphi \psi + i h a_{02} \left\{ \frac{1}{2} \frac{\partial^2 \varphi}{\partial t^2} + \frac{i}{\hbar} e \varphi \frac{\partial^2 \psi}{\partial t^2} + \frac{i}{2 \hbar} e \psi \frac{\partial \varphi}{\partial t} - \frac{1}{2 \hbar^2} e^2 \varphi^2 \psi \right\} + i h \frac{1}{6} a_{30} \left\{ \Delta \psi - \frac{ie}{\hbar c} (\nabla \cdot A) \psi - \frac{i e^2}{\hbar c^2} (A \cdot \nabla) \psi - \frac{e^2}{\hbar c^2} (A \cdot A) \psi \right\},$$

where the $a_{ij}$ denote the parts of the ratios $I_{ij}/I_0$, of the integrals that are independent of $\tau$:

$$I_{ne} = \int_0^\infty \int_0^{ct} \exp \left( -i \frac{m_0 e^2}{\hbar} t \sqrt{1 - \left( \frac{r}{ct} \right)^2} \right) f(r, ct) r^{n+2} dr dt, \quad (n = 0, 2; \kappa = 0, 1, 2)$$

In the general case $I_{nk}$ are certain rapidly varying functions. However, when neglecting microtime fluctuations in the probability amplitude, the $a_{ij}$ can be estimated without going into the weight function $f(s)$ in too much detail. We represent $f(s)$ as a certain sum

$$f(s) = \sum_{m=1} P_m s^m,$$

where $P_m$ and $\gamma_m$ are constants with $\gamma_m > 0$. Then expanding (5) in a Taylor series in powers of $(r/(ct))^2$, we obtain after substitution of $f(s)$ into (4)

$$I_{nk} = \int_0^\infty \sum_{l=0} f_l(ct) \left( \frac{1}{n+2l+3} - \frac{V}{2}(\frac{2}{at})^{\frac{n+2l+3}{2}} \Gamma\left( \frac{n+2l+3}{2} \right) \right) \times$$

$$\times H_{n+2l+4}(at) \right) dt - ic^{n+3} \int_0^\infty \sum_{l=0} f_l(ct) \left( \frac{V}{2}(\frac{2}{at})^{\frac{n+2l+3}{2}} \Gamma\left( \frac{n+2l+3}{2} \right) J_{n+2l+4}(at) \right) dt,$$

where $a = m_0 e^2/\hbar$; $J_v(z)$ and $H_v(z)$ are the Bessel and Struve functions, respectively.

To evaluate (7), we assume $< > \gg (m_0 e^2)^{-1}$, and use the known representation [4] of $J_v(z)$ and $H_v(z)$ in terms of trigonometric functions. We hence take into account that

$$\int_0^1 \sin(at) t^{\nu-1} dt = -\frac{1}{2\nu} F_1(\nu; \nu + 1; -i a);$$

$$\int_0^1 \cos(at) t^{\nu-1} dt = \frac{1}{2\nu} [F_1(\nu; \nu + 1; i a) + F_1(\nu; \nu + 1; -i a)]$$

($F_1(a; b; z)$ is the degenerate hypergeometric function). Then after a number of manipulations by making the assumption that the minimal value of $\gamma_m$ exceeds 0.25, we obtain to the accuracy of higher order infinitesimal terms:

$$I_{10} = -c^2 \cdot 1.5 V \pi^{3/2} a^{-7/2} f_0(\alpha \tau) \times \{[\sin(\alpha \tau) - \cos(\alpha \tau)] + i[\sin(\alpha \tau) + \cos(\alpha \tau)]\};$$

$$I_{02} = c^2 \cdot 0.5 V \pi^{3+1/2} a^{-5/2} f_0(\alpha \tau) \times \{[\sin(\alpha \tau) - \cos(\alpha \tau)] + i[\sin(\alpha \tau) + \cos(\alpha \tau)]\} \times \{i + [\kappa + 27/8 - 5f_1(\alpha \tau)]/f_0(\alpha \tau)\}(\alpha \tau), \quad (\kappa = 1, 2).$$

Without specifying the values of the parameters in (5), the relationships (10) and (11) permit estimation of the factors in front of the braces in (3) and writing of the equation.