It is shown that the relativistic constraint of the length of possible trajectories in the method of integrals over the trajectories results in the Klein–Gordon equation.

In [1, 2], it was proposed to introduce a weight function $t_{0}^{n_{0}}$, into account the probability of realizing a trajectory in taking account of the contribution of each trajectory in the probability amplitude in the method of integrals over Feynman trajectories [3]. Such trajectory separation results [2] in quasirelativistic corrections to the Schroedinger equation. In this paper the influence of the weight function is examined in a completely relativistic approximation.

The integral equation for the $\psi$-function that is invariant with respect to the Lorentz transformation has the following form for a charged particle with zero spin in an electromagnetic field:

$$
\psi(r_{2}, t_{2}) = \int_{t_{1}}^{t_{0}} K(r_{2}, t_{2}; r_{1}, t_{1}) \psi(r_{1}, t_{1}) d\tau \, dt_{1},
$$

where, under the condition of smallness of the parameter $\tau = t_{1} - t_{0}$, the propagator equals

$$
K(r_{2}, t_{2}; r_{1}, t_{1}) = B \exp\left\{-\frac{i}{\hbar} m_{0} c \frac{(t_{2} - t_{1})}{t_{1}} \sqrt{1 - \frac{(r_{2} - r_{1})^2}{c^2(t_{2} - t_{1})^2}} - \frac{i}{\hbar} \frac{e}{c} \frac{r_{2}^{2} + r_{1}^{2}}{3} \frac{t_{2} + t_{1}}{2} + i \frac{e}{\hbar} \frac{r_{2} - r_{1}}{c} \frac{t_{2}^{2} + t_{1}^{2}}{2} \right\} \times f(r_{2} - r_{1}, t_{2} - t_{1}).
$$

$m_{0}$ is the particle rest mass; $e$ is its charge, $c$ is the velocity of light, $\varphi$ and $A$ are the scalar and vector field potentials, $B$ is a normalizing constant (its value is determined [3] by the rapidly varying part of the propagator), $f(r_{2} - r_{1}, t_{2} - t_{1})$ is a real function of the interval $s = (c^{2}(t_{2} - t_{1})^{2} - (r_{2} - r_{1})^{2})^{1/2}$ that takes account of the weight of each possible trajectory, and it is assumed that $\lim_{s \to 0} f(s) = 0$ (such an assignment of $f(s)$ automatically takes into account that the trajectory length should be less than $c(t_{2} - t_{1})$ for a particle with non-zero rest mass).

Using the smallness of $\tau$, we go from the integral to a differential equation. To do this, we expand

$$
\exp\left\{-\frac{i}{\hbar}(t_{2} - t_{1}) e^{\varphi} \left(\frac{r_{2} + r_{1}}{2}, \frac{t_{2} + t_{1}}{2}\right) + i \frac{e}{\hbar} \frac{r_{2} - r_{1}}{c} \frac{t_{2}^{2} + t_{1}^{2}}{2} \right\}
$$

and the wave function under the integral sign in series in $t = t_{2} - t_{1}$ and $r = r_{2} - r_{1}$ to the accuracy of third order infinitesimals and in conformity with the general methodology [3] (after determining the normalizing constant), we extract components independent of $\tau$. Then after integration over the angles we obtain the expression

$$
\hbar \frac{\partial \psi}{\partial t} = e \varphi \psi + i \hbar a_{0} \left\{ \frac{1}{2} \frac{\partial \varphi}{\partial t^{2}} + \frac{i}{\hbar} e \frac{\partial \varphi}{\partial t} + \frac{e}{2 \hbar} \frac{\partial \varphi}{\partial t} - \frac{1}{2 \hbar} e^{2} \varphi \frac{\partial \varphi}{\partial t} \right\} + i \hbar \frac{1}{6} a_{30} \left( \Lambda \varphi - \frac{ie}{\hbar c} (\nu \cdot A) \psi - \frac{i 2e}{\hbar c} (A \cdot \nu) \varphi - \frac{e^{2}}{\hbar^{2} c^{2}} (A \cdot A) \varphi \right),
$$

where the $a_{ij}$ denote the parts of the ratios $I_{ij}/I_{01}$ of the integrals that are independent of $\tau$:

$$I_{ne} = \int_0^\pi \int_0^{ct} \exp \left( -i \frac{m_0 \alpha^2}{h} t \sqrt{1 - \frac{r}{ct}} \right) f(r, ct) r^{n+2} dr dt,$$

$$n = 0, 2; \kappa = 0, 1, 2$$

In the general case $I_{nk}$ are certain rapidly varying functions. However, when neglecting microtime fluctuations in the probability amplitude, the $a_{ij}$ can be estimated without going into the weight function $f(s)$ in too much detail. We represent $f(s)$ as a certain sum

$$f(s) = \sum_{m=1}^\infty P_m s^{2m},$$

where $P_m$ and $\gamma_m$ are constants with $\gamma_m > 0$. Then expanding (5) in a Taylor series in powers of $(r/(ct))^2$, we obtain after substitution of $f(s)$ into (4)

$$I_{nk} = \int_0^\pi \sum_{l=0}^{n+3} f_l(ct) \left( \frac{1}{n+2l+3} - \frac{\sqrt{\pi}}{2} \left( \frac{2}{at} \right)^{n+2l+3} \Gamma \left( \frac{n+2l+3}{2} \right) \right) \times$$

$$\times H_{n+2l+4}(at) \right) dt - \text{ic}^{n+3} \sum_{l=0}^{n+3} f_l(ct) \left[ \frac{\sqrt{\pi}}{2} \left( \frac{2}{at} \right)^{n+2l+2} \Gamma \left( \frac{n+2l+3}{2} \right) J_{n+2l+4}(at) \right] dt,$$

where $a = m_0 \alpha^2/h$; $J_\nu(z)$ and $H_\nu(z)$ are the Bessel and Struve functions, respectively.

To evaluate (7), we assume $\gamma > \frac{h}{m_0 \alpha^2}$, and use the known representation [4] of $J_\nu(z)$ and $H_\nu(z)$ in terms of trigonometric functions. We hence take into account that

$$\int_0^1 \sin(at) t^{\nu-1} dt = -\frac{i}{2\nu} \left[ F_1(\nu; \nu + 1; ia) - iF_1(\nu; \nu + 1; -ia) \right],$$

$$\int_0^1 \cos(at) t^{\nu-1} dt = \frac{1}{2\nu} \left[ F_1(\nu; \nu + 1; ia) + F_1(\nu; \nu + 1; -ia) \right]$$

($F_1(a; b; z)$ is the degenerate hypergeometric function). Then after a number of manipulations by making the assumption that the minimal value of $\gamma_m$ exceeds 0.25, we obtain to the accuracy of higher order infinitesimal terms:

$$I_{10} = -c^3 \cdot 1.5 \sqrt{\pi/2} a^{-5/2} f_6(ct) \times \{ [\sin(a\tau) - \cos(a\tau)] + i \{ \sin(a\tau) + \cos(a\tau) \} \};$$

$$I_{01} = c^3 \cdot 0.5 \sqrt{\pi/2} a^{-5/2} f_6(ct) \times \{ [\sin(a\tau) - \cos(a\tau)] + i \{ \sin(a\tau) +$$

$$+ \cos(a\tau) \} \} \times \{ i + \{ \kappa + 27/8 - 5f_1(ct)/f_6(ct) \} / (a\tau) \} \};$$

$$\kappa = 1, 2.$$