and
\[ Y(p, q, k) = [(s + 1)(s + c)(s + p + 1)]^{-1} F_3 \left[ \begin{array}{c} 1, s + a + 1, s + b + 1, s + p + 1; 1 \\ \sigma + 2, \sigma + c + 1; \rho + \sigma + 2; \end{array} \right]. \]

Here we have used the well-known result [8]
\[ \int_0^1 dy y^{\sigma-1} F_3(a, b, c; y) = \beta (\beta + c - 1)(\beta + a) F_3 \left[ \begin{array}{c} 1, \beta + a, \beta + b, \beta + c; 1 \\ \beta + 1, \beta + c, a + \beta + 1; 1 \end{array} \right]. \]

LITERATURE CITED

ORTHOGONAL POLYNOMIALS IN THE LIE ALGEBRAS
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We present a method of constructing orthogonal polynomials generated by pairs of Hermitian operators in representations of Lie algebras. All known classical polynomials of both discrete and continuous argument are generated naturally by the simplest Lie algebras.

1. Introduction

The group-theoretic approach to the theory of special functions of a continuous argument is well known [1]. In recent years a connection between special functions of a discrete argument (the Clebsh–Gordan coefficients, the Kravchuk polynomials) and representations of Lie groups and algebras has been suggested [2, 3].

In the present paper we give a general method of constructing polynomials of discrete and continuous argument and show how this method works by considering examples of the simplest Lie algebras. The key to the method is the identification of pairs of Hermitian operators in representations of the Lie algebra, where one of the operators has a discrete spectrum and generates the order of the polynomial, and the other is tridiagonal in the base set of the first operator and its eigenvalues serve as the argument of the polynomial. The weighting functions are expressed in terms of "vacuum" matrix elements.

It turns out that the representations of the simplest Lie algebras generate in a natural way all known classical polynomials of both discrete and continuous argument. This allows one to greatly simplify the theory of polynomials of discrete argument (recurrence and difference equations, generating functions, addition theorems, and so on) by expressing them in terms of...
the language of the corresponding algebra or group, as was done by Vilenkin [1] for certain special functions.

Our approach unifies various separate results on the relation between polynomials of discrete argument and Lie groups [2, 3].

2. General Method

We consider a unitary representation of a Lie algebra in which the Hermitian operator $K$ has a discrete spectrum in the base set $\Psi_k$:

$$K\Psi_k = \lambda_k \Psi_k, \ k = 0, 1, \ldots . \quad (2.1)$$

We introduce a second operator $P$ which is tridiagonal in this base set

$$P\Psi_k = \Psi_{k+1} + \alpha_k \Psi_k + \alpha_k \Psi_{k-1}. \quad (2.2)$$

Its spectrum

$$p\Psi_p = \mu_p \Psi_p \quad (2.3)$$

can be either discrete or continuous.

We write the matrix element $U_{pk} = \langle \Psi_p | \Psi_k \rangle$ in the form

$$U_{pk} = \langle \Psi_p | \Psi_0 \rangle \Pi_k(\mu_p). \quad (2.4)$$

Then we have from (2.2) and (2.3):

$$z_{k+1} \Pi_{k+1}(\mu_p) + (\alpha_k - \mu_p) \Pi_k(\mu_p) + \alpha_k \Pi_{k-1}(\mu_p) = 0, \quad (2.5)$$

and

$$\Pi_{-1}(\mu_p) = 0, \quad \Pi_0(\mu_p) = 1. \quad (2.6)$$

The conditions (2.5) and (2.6) uniquely determine a system of orthogonal polynomials $\Pi_k$ of order $k$ and argument $\mu_p$. The orthonormality of these polynomials follows from the completeness of the system $\Psi_p$ and the orthonormality of the base functions $\Psi_k$:

$$\sum_p \omega_p \Pi_k(\mu_p) \Pi_{k'}(\mu_p) = \delta_{kk'}, \quad (2.7)$$

where the weight has the form

$$\omega_p = |\langle \Psi_p | \Psi_0 \rangle|^2. \quad (2.8)$$

In many cases the state $|\Psi_p\rangle$ is obtained from $|\Psi\rangle$ by the operation of a unitary operator $U$ of the corresponding group. Therefore the weight will be the probability distribution in the coherent state (in the sense of Perelomov [4]). In the case of a continuous spectrum, the summation in (2.7) is replaced by an integration with respect to $\mu$.

We note that the method of constructing theories of orthogonal polynomials with the help of tridiagonal (Jacobian) matrices is well known and goes back to Chebyshev [5]. Our approach differs from the classical methods in two ways: the Lie algebra is used to construct the characteristics of the polynomials, and the "coherent" factor $\langle \Psi_p | \Psi_0 \rangle$ in (2.4), which determines the weighting function (2.8), is identified explicitly.

3. $SU(1, 1)$ Algebra. The Meixner, Laguerre, and Polacheck Polynomials

The method described in the preceding section is nicely illustrated with the example of the algebra $SU(1, 1)$, which is generated by the three generators:

$$K_0, \ K_{\pm} = K_1 \pm K_2, \quad [K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_-, K_+] = 2K_0. \quad (3.1)$$

We consider a discrete series representation, in which the operator $K_0$ is diagonal:

$$K_0\Psi_k = (\nu + \kappa)\Psi_k, \ \nu > 0, \ \kappa = 0, 1, \ldots . \quad (3.2)$$