FINITE PLANE DEFORMATIONS OF A COMPRESSIBLE MATERIAL

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A problem on the finite plane deformations of a solid body is formulated without placing limitations on the displacements and deformations. The dependence of the hydrostatic stress on volume changes and the deformation law are both arbitrary. The coordinates of the points of the body in the initial state are taken as the basic coordinates. The problem is reduced to the solution of a nonlinear equation for the generalized stress function.

§1. Geometric characteristics of plane deformations.

Plane deformation in Cartesian coordinates is defined by specifying the displacements \( u_k \) (\( k = 1, 2 \)) as functions of the coordinates \( x_k \) in the initial state. The components of the Cauchy tensor and the rotation components are given by the formulas

\[
\begin{align*}
\varepsilon_{11} &= u_{1,1}, \\
\varepsilon_{22} &= u_{2,2}, \\
2\varepsilon_{12} &= u_{1,2} + u_{2,1}, \\
\varepsilon_{33} &= \varepsilon_{23} = \varepsilon_{13} = 0, \\
2\alpha_3 &= -u_{1,2} + u_{2,1}.
\end{align*}
\]

We already know [4] the expressions for the elongations \( \lambda_{SS} \) of the coordinate fibers, the coefficients of distortion of the areas of the coordinate surface elements \( s_k \), and the projections \( \alpha_{nk} \) of the vectors characterizing the orientations of the coordinate surface elements following deformation in terms of \( \varepsilon_{ik} \) and \( \alpha_{i} \).

We shall characterize the plane deformed state by means of four other independent parameters with a straightforward geometric significance: the angle \( \vartheta \) between the \( x_1 \) axis and the first coordinate direction, the angle of rotation \( \omega \) of the principal fibers in the deformed state, and the principal elongations \( \lambda_1, \lambda_2 \).

In addition to the principal deformations, we introduce the two following symmetrical deformation invariants: the relative volume change \( \Delta \) and the deformation intensity \( \vartheta \). Setting

\[
\begin{align*}
\lambda_1^2 &= (1 + \Delta) \exp \vartheta, \\
\lambda_2^2 &= (1 + \Delta) \exp (-\vartheta),
\end{align*}
\]

we readily obtain

\[
\begin{align*}
1 + \varepsilon_{11} &= \sqrt{1 + \Delta} [\cosh 0.5 \vartheta \sin 0.5 \vartheta \cos (2\vartheta + \omega)], \\
\varepsilon_{22} - \alpha_3 &= \sqrt{1 + \Delta} [-\cosh 0.5 \vartheta \sin 0.5 \vartheta \cos (2\vartheta + \omega)], \\
1 + \varepsilon_{22} &= \sqrt{1 + \Delta} [\cosh 0.5 \vartheta \sin 0.5 \vartheta \cos (2\vartheta + \omega)], \\
\varepsilon_{12} + \omega_3 &= \sqrt{1 + \Delta} [\cosh 0.5 \vartheta \sin 0.5 \vartheta \cos (2\vartheta + \omega)].
\end{align*}
\]

From these equations we obtain the compatibility equation for the deformation parameters,
and the deformation intensity
\[
\frac{4}{\rho} \mathbf{T}^2(s) = 4U_{12}^2 + (U_{zz} - U_{11})^2.
\] (2.5)

We can now determine the relative volume change by using the third relation of (2.3).

The compatibility condition for the deformation parameters (1.4) provides an equation for the determination of the generalized stress function,
\[
pG(U_{11} - U_{12})_{11} + pG(U_{11} - U_{12})_{11} +
+ 2G(\mathbf{U})_{12} + 4p\left(\mathbf{G}U_{12} + \frac{\Delta_1}{1 + \Delta} \mathbf{G} (U_{11} - U_{12})_{11} +
+ 2\mathbf{U} (\mathbf{v} + \mathbf{w}) + \mathbf{v} \cdot \mathbf{w}
\right)_{12} +
\] (3.1)

we use formulas (2.3) to find expressions for the generalized shearing stress \( \mathbf{T} \) and for the angle \( \theta \),
\[
\mathbf{T} = 4pU_{12}J_{12}, \quad 4\theta = \ln U_{zz} - \ln U_{zr},
\] (3.2)
as well as the expression relating the relative volume change, deformation intensity, and stress function,
\[
(1 + \Delta ) \sigma - \frac{\Delta_1}{\Delta} \mathbf{od}\Delta - \frac{\mathbf{v}}{\Delta} \mathbf{T} = 2pU_{zz}.
\] (3.3)

The resolvent for the problem can be written as
\[
2 \text{Re} \left\{ \left( \mathbf{G}U_{zz} \right)_{zz} + \left[ \frac{1}{1 + \Delta} \mathbf{U}_{zz} + \frac{1}{2} \mathbf{U}_{zz} \right]_{zz} +
+ \left( \mathbf{v} \cdot \mathbf{w} \right)_{zz} + \mathbf{v} \cdot \mathbf{w}
\right\} = 0.
\] (3.4)

Expressions (3.2) and (3.3), together with the physical relations (2.2), allow us to assert that Eq. (3.4) contains only the one unknown function \( U \).

§4. Special forms of the resolvent. The volume change law can be written as
\[
\sigma = K \left[ \frac{\Delta}{1 + \Delta} \right],
\] (4.1)
where \( K \) is the volume deformation modulus. Using the formula
\[
(1 + \Delta ) \sigma - \frac{\Delta_1}{\Delta} \mathbf{od}\Delta = K \ln (1 + \Delta)
\] and expression (3.3) we obtain
\[
\ln (1 + \Delta) = \frac{1}{K} \left( 2pU_{zz} + \frac{\mathbf{v}}{\Delta} \mathbf{T} \right).
\] (4.2)

Since in this case
\[
\frac{\Delta}{1 + \Delta} = \frac{1}{K} \left( 2pU_{zz} + \frac{\mathbf{v}}{\Delta} \mathbf{T} \right),
\]
the resolvent becomes
\[
2p \text{Re} \left\{ \left( \mathbf{v} \cdot \mathbf{w} \right)_{zz} + \frac{\mathbf{v}}{\Delta} \mathbf{T} \right\} = 0.
\] (4.3)

Let us investigate the deformation laws corresponding to "soft" materials,
\[
T = 2g \mathbf{v} \cdot \mathbf{w},
\] (4.4)
and to "hard" materials,
\[
T = 2g \mathbf{v} \cdot \mathbf{w},
\] (4.5)
where \( g \) is the generalized shear modulus. Expressing the stress function \( U \) as a power series in terms of a small parameter \( \mu \),
\[
U = \mathbf{U} + \mu \mathbf{U}_{\mathbf{U}} + \mu^2 \mathbf{U}_{\mathbf{U}^2} + \ldots, \quad \mu = \frac{p}{2g},
\] (4.6)
and confining ourselves to powers of \( \mu \) lower than the third, we obtain the differential equations for successive orders of approximation:
\[
L_0 = 0, \quad L_1 = -\frac{1 + 2h}{2(1 + h)} \left( \mathbf{U}_{zz} \mathbf{U}_{zz} + 2 \text{Re} \left( \mathbf{U}_{zz} \mathbf{U}_{zz} \right) \right),
\] (4.7)
\[
L_2 = -\frac{2(1 + 2h)}{1 + h} \text{Re} \left( \mathbf{U}_{zz} \mathbf{U}_{zz} \right) + \frac{\nu + h}{1 + 2h} \mathbf{U}_{zz} \mathbf{U}_{zz}.
\] (4.8)

Here \( \nu = 1 \) corresponds to the law (4.4) and \( \nu = 0 \) to the law (4.5).

The solutions of equations (4.7) and (4.8) can be represented as sums of particular solutions \( \mathbf{U}_{k, \mathbf{U}} \) and known combinations of the arbitrary analytic functions \( \mathbf{v}_k, \mathbf{v}_k \mathbf{v}_k \mathbf{v}_k \) [2],
\[
\text{Re} \mathbf{U} = \text{Re} \left( \mathbf{U} + \mathbf{v}_k + \mathbf{v}_k \right), \quad k = 0, 1, 2, \ldots.
\] (4.9)