CAUSTICS OF SPACE-TIME FOLIATION

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It is shown that the gravitational singularities described as space-time foliation singularities are caustics and topological reconstructions.

The criterion of gravitational singularities continues to be a subject of discussion [1, 2]. It was proposed to describe gravitational singularities as space-time foliation singularities [4-6], as an alternative to the known Penrose-Hocking-Ellis b-criterion [3] and which would, in contrast to the b-criterion, permit establishment of the singularity structure. It is shown in this paper that the space-time foliation singularities are topological reconstructions and caustics. Our approach is based on the following theorems.

**THEOREM 1.** For any gravitational field $g$ there exist a 1-form $\omega$ and a Riemannian metric $g^R$ in the manifold $X^4$ such that

$$g = g^R - 2\omega \otimes \omega/|\omega|^2, \tag{1}$$

where $|\omega|^2 = g^R(\omega, \omega) = -g(\omega, \omega)$ be a non-singular 1-form on $X^4$. For any Riemann metric $g^R$ on $X^4$ a pseudo-Riemann metric $g$ exists such that the relationship (1) is satisfied. The form $\omega/|\omega|$ agrees with the tetrade form $h^\mu = h^\mu_{\nu}dx^\nu$ of the gravitational field $g$.

**THEOREM 2.** A mutually one-to-one correspondence exists between the nonsingular forms $\omega$ on $X^4$ and smooth orientatable distributions $F$ of 3-dimensional subspaces of the tangential spaces in $X^4$ which are determined by the equation $\omega(F) = 0$.

The distribution $F$ is called space-time if it is generated by a tetrade of the form $\omega = h^0_{\nu}$ of a certain gravitational field.

**COROLLARY.** Any gravitational field determines the space-time distribution on the manifold $X^4$. Conversely, every orientable 3-dimensional distribution on $X^4$ is space-time relative to a certain gravitational field.

The space-time distribution $F$ determines the space-time structure on the manifold $X^4$ consistent with $g$, and a corresponding Riemann metric $g^R$ is a locally Euclidean topology on $X^4$ consistent with $g$.

Let us call a space-time structure causal if the space-time distribution $F$ is integrable, i.e., $\omega \wedge d\omega = 0$, and its generating form is exact, i.e., $\omega = df$. In this case foliation of the manifold $X^4$ into hypersurfaces holds such that hyperplanes of the distribution $F$ are tangents to the layers of this foliation. Such a causality condition is equivalent to stable Hocking causality [3]. Layers of a causal foliation are level surfaces of the generating functions $f$ and some curve transversal to the foliation layers does not intersect any layer more than once.

The stable causality condition assures the absence of any disturbances of causality in space-time. Consequently, taking account of the correspondence between the gravitational fields and the space-time distributions, the following criterion of gravitational singularities was proposed.

It is considered that a gravitational field $g$ on a manifold $X^4$ contains no singularities if it allows a causal space-time foliation $F$ consistent with them and a complete Riemann metric $g^R$ on $X^4$.

Three kinds of singularities can be distinguished in conformity with this criterion. Among the first kind are gravitational fields characterizing the singularity of the Riemann metric $g^R$ on $X^4$. This means that the topology of the manifold on $X^4$ is not consistent with
the topology governed by a gravitational field by means of the metric \( g^R \). If points of such
singularities are excluded from space-time, the remaining space can be transformed into a
complete Riemann space without singularities by conformal mapping of the metrics \( g^R \) and \( g \).
Conformal mapping does not influence the space-time distribution, consequently, singularities
of the first kind are conformal singularities.

The second kind includes gravitational fields that allow a Riemann metric and a space-
time distribution but not a causal foliation. Such fields being regular in themselves are
accompanied by disturbances of causality of space-time [7]. Let us emphasize the global na-
ture of such disturbances since every gravitational field locally governs the causal foliation.

Among the third kind of gravitational singularities that we have collected here for in-
vestigation are gravitational fields that do not allow regular space-time distributions.
Speaking graphically, these singularities occur at point of intersection of integral lines
(time lines) of the vector field \( h_0 \) dual to the form \( h^0 \) and orthogonal to layers of the
distribution \( F \). Consequently, the idea of describing such singularities is to raise the
distribution \( F \) into the total space \( tL(T(X^v)) \) (or \( tL^2(T(X^v)) \) of the tangential (or cotangential)
stratification, to continue it there along the line of the raised field \( h_0 \) beyond the point
of intersection of \( h_0 \) in \( X^v \) and to project again on \( X^v \). The critical points of this projec-
tion will correspond to singularities of the distribution \( F \). Locally (in terms of sprouts)
such singularities can be represented as singularities of causal foliation and their descrip-
tion is based on the following theorem.

**THEOREM 3.** For any foliation of the level surfaces \( F \) in the manifold \( X^n \) there exists a
foliation of level surfaces \( F' \) on a Lagrange submanifold of the total space \( tL^2(T(X^v)) \) such
that \( F \) is an image of \( F' \) relative to Lagrange mapping. The foliation \( F' \) itself is here in-
duced on a Lagrange submanifold by the foliation \( \{R \exists z = \text{const} \} \) on \( tL^2(T(X^v)) \times R \).

Let us briefly recall the following mathematical constructions, the Hefflinger struc-
tures and caustics.

A smooth mapping \( \varphi: Y \rightarrow X \) of the manifold \( Y \) into the manifold \( X \) with the foliation \( F \) is
called transversal \( F \) if \( T_X(X) = T_X(F) + \text{Im} d\varphi_X \) at all points \( x \in X \). When the mapping \( \varphi \) is
transversal to \( F \), the prototypes of the layers of \( F \) form an induced foliation \( \varphi^*F \) on the
manifold \( Y \) and \( \text{codim} \varphi^*F = \text{codim}F \). If the mapping is not transversal to the foliation \( F \) on \( X \),
the induced construction also has a definite geometric meaning (foliation with singularities)
and is called a Hefflinger structure [8]. We limit ourselves to the case when the Hefflinger
structure on the manifold \( X \) is generated by a real function \( F \) (induced by the imbedding
\( X \ni x \rightarrow (x, j(x)) \in X \times R \), where the product \( X \times R \) is allotted to the foliation \( \{R \exists z = \text{const} \} \)
having critical points, i.e., points where \( df = 0 \). The level surfaces of such a function
form a partition (but not foliation) of the manifold \( X \). The topology of the layers of this
partition generally varies during passage through the level surface containing a critical
point. An example is the topological reconstruction of level surfaces of the Morse function
described by the following theorem, at the critical points [9].

**THEOREM 4.** Let \( f \) be a Morse function on the manifold \( X^n \). Let \( M_+ \) and \( M_- \) denote level
surfaces of \( f \) before and after the critical point \( x_0 \) of index \( k \). There exists a \( k \)-dimen-
sional cell \( e^k \) and an \( (n - k) \) dimensional cell \( e^{n-k} \) such that \( e^k \cap e^{n-k} = x_0 \), \( M_- \cap e^k = \partial e^k \), \( M_+ \cap e^{n-k} = \partial e^{n-k} \) and \( M_- - \partial e^{n-k} \) differomorphic to \( M_+ - \partial e^k \) (\( \partial \) denotes the boundary).

By analogy with geometric optics, caustics in the theory of gravitation are the geo-
metric locus of focal and conjugate points of congruence of geodesics [10, 11]. In our case,
however, the field of orthonormals to the foliation in the general case is not geodesic and
we use the general mathematical definition of the caustics as Lagrange mapping singularities [12].

The description of every caustic locally (in terms of sprouts) can be reduced to the
following standard form. Let the space \( R^{2n} \) be provided with the coordinates \( (x^1, P_j) \).
Let us consider the 1-form \( \alpha = P_j dx^j \) and the submanifold \( N \subset R^{2n} \) such that \( dx(N) = 0 \), i.e., the
form \( \alpha \), being bounded on the submanifold \( N \), becomes exact \( \alpha(N) = dz(N) \). Such a manifold of
the maximal dimensionality \( n \) is called Lagrangian. A Lagrangian manifold can be constructed
by using the generating function \( S(x^i, P_j) \) of \( n \) variables \( \{x^i, P_j, i \in I, j \in J \} \) (where \( I, J \)
is a certain partition of the set \( 1, \ldots, n \)). It is defined by the following relationships
\[
\frac{\partial S}{\partial x^i} = -\frac{\partial S}{\partial P_j}, \quad P_j = \frac{\partial S}{\partial x^i}.
\]
Let \( \tau: \{x^1, P_j\} \rightarrow \{x^1\} \) be the projection of \( R^{2n} \) on \( R^n \). This projection, bounded on the
Lagrange manifold \( N \)

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