ACCURATE SOLUTIONS OF QUASIPOTENTIAL EQUATIONS
WITH ORBITAL MOMENTUM $s = 1$

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It is shown that the integral quasipotential Logunov-Tavkhelidze and Kadyshevskii equations with local - in Lobachevskii momentum space - quasipotentials, the transforms of which are even rational functions of $r$ in relativistic configurational space, may be reduced to a Sturm-Liouville problem in the case of unit orbital momentum. In the critical limit, when the bound-state mass is zero, accurate wave functions are obtained.

The quasipotential approach in quantum field theory [1-3] is widely used in describing the dynamics of elementary-particle interaction. In this connection, the investigation of methods of solving integral quasipotential equations is an urgent problem. One such method is based on reducing the integral equations to differential form directly in the momentum representation [4-7].

In [4, 5], quasipotentials of the form $V(p, \kappa) = V(p - \kappa)$ were used; these are Fourier transforms - in the sense of the usual expansion of $\exp(i pr)$ in plane waves - of rational functions that are even in ordinary $r$-space. Expansion of unitary irreducible Lorentz-group representations in terms of the matrix elements [8] allows the quasipotential equations to be written in a relativistic configurational representation [9, 10], where they take the form of difference equations. The relativistic coordinate here is conjugate with the velocity rather than the momentum, and the potentials are local in three-dimensional Lobachevskii momentum space realized in the upper field of the hyperboloid $p^2 - m^2 = 0$ [10]. In [6, 7], the class of such potentials and the case of spherically symmetric wave functions was investigated. Reducing the integral equations to differential form allows their solution to be found in the chiral limit (zero bound-state mass).

In the present work, it is shown that, for the potentials in [6, 7], the integral equations may be reduced to differential form in the case of wave functions with an angular dependence corresponding to unit orbital (and total) momentum. In the chiral limit, their accurate solutions are found.

The quasipotential equations for the wave functions of relative motion of spinless particles are written as follows [1-3]

$$ G^{-1}_i(E, p_0) \psi(p) = (2\pi)^{-3} \int V(p, \kappa; E) \psi(\kappa) m d\kappa / k_0. \quad (1) $$

The inverse free Green's functions $G^{-1}_i$ corresponding to the Logunov-Tavkhelidze ($i = 1$) equation [1] and the Kadyshevskii ($i = 2$) equation [3] take the form (in the case of equal particle masses $m_1 = m_2 = m$)

$$ G^{-1}_1(E, p_0) = E^2 - p_0^2; \quad G^{-1}_2(E, p_0) = p_0 (E - p_0). \quad (2) $$

In Eq. (1), $2E$ is the bound-state mass; the particle momenta belong to the mass surface $p_0 = (p^2 + m^2)^{1/2}$; $\kappa_0 = (\kappa^2 + m^2)^{1/2}$; the quasipotential $V$ is determined outside the energy surface $p_0 = k_0 = E$.

The local - in three-dimensional space - Lobachevskii quasipotential depends on the non-Euclidean difference vector $\Delta_{p, \kappa} = p(-) \kappa$ of two momenta: $V(p, \kappa; E) = V(\Delta_{p, \kappa})$ where

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\[ \Delta_{\rho,\kappa} = \Lambda_{\kappa}^{-1} \rho = \rho - \frac{\kappa}{m} \left[ p_0 - \frac{p\kappa}{\kappa_0 + m} \right]; \quad \Delta_{\rho,\kappa}^0 = (\Lambda_{\kappa}^{-1} \rho)^0 = (m^2 + \Delta_{\rho,\kappa}^0)^{1/2}. \]

Here \( \Lambda_{\kappa}^{-1} \) is the pure Lorentz transform such that \( \Lambda_{\kappa}^{-1}(\kappa_0, \kappa) = (m, 0) \).

Consider a quasipotential specified in the following form in relativistic configurational and momentum space \([6, 7]\)

\[ V(r) = -\frac{g^2}{r^2}; \quad V(\Delta_{\rho,\kappa}) = -\frac{2\pi^2 g^2}{|\Delta_{\rho,\kappa}|}. \]

Expanding \( V(\Delta_{\rho,\kappa}) \) in spherical harmonics

\[ (2\pi)^{-3} V(\Delta_{\rho,\kappa}) = \sum_{n=0}^{\infty} \sum_{\mu=-n}^{n} (p\kappa)^{-1} V_n(p, \kappa) Y_{n\mu}(n_\rho) Y_{n\mu}^*(n_\kappa). \]

where \( \kappa = |\kappa|, \ n_\kappa = \kappa/\kappa, \) etc., the partial potential \( V_n(p, k) \) is written in the form

\[ V_n(p, \kappa) = (2\pi)^{-2} p\kappa \int \frac{1}{r} V(\Delta_{\rho,\kappa}) P_n(\cos \theta_{p,\kappa}) d \cos \theta_{p,\kappa}. \]

The wave function \( \Psi(p) \) is sought in the form

\[ \Psi(p) = \psi(p) Y_{l\mu}(n_p). \]

Substituting Eqs. (4) and (6) into Eq. (1) and introducing the function \( F_1(p) = G_{\rho}^{-1}(E, p_0) \rho \psi(p) \), the following one-dimensional integral equation is obtained

\[ F_1(p) = \int_0^\infty V_1(p, \kappa) G_1(E, \kappa_0) F_1(\kappa) m d\kappa/\kappa_0. \]

This equation is now considered in the case when \( \ell = 1 \) (for \( \ell = 0 \), see \([6, 7]\)) for the potential in Eq. (3), which takes the following form according to Eq. (5)

\[ V_1(p, \kappa) = -g^2 m[\theta(\chi_\rho - \chi_\kappa) u(\chi_\rho) v(\chi_\kappa) + \theta(\chi_\kappa - \chi_\rho) u(\chi_\kappa) v(\chi_\rho)]. \]

The velocities \( \chi_p \) and \( \chi_k \) corresponding to momenta \( p \) and \( k \) are determined by the relations

\[ p_0 = m\cosh \chi_p, \quad \kappa_0 = \tilde{m} \cosh \chi_k \]

and explicit form of the functions \( u \) and \( v \) is as follows

\[ u(\chi) = \cosh \chi; \quad v(\chi) = \chi \sinh \chi - 1. \]

It is important here that the following properties hold

\[ u''(\chi) = -2u(\chi) v(\chi); \quad v''(\chi) = v(\chi) u(\chi); \]

\[ \beta(\chi) = 2\sinh^2 \chi; \quad u(\chi) v'(\chi) - u'(\chi) v(\chi) = 1. \]

Using Eq. (10), it is simple to show that the integral Eq. (7) with the partial potential in Eq. (8) is equivalent to the Sturm-Liouville problem consisting of the differential equation

\[ -F_1''(\chi) + \beta(\chi) F_1(\chi) = m^2 g^2 G_1(E, m \cosh \chi) F_1(\chi) \]

and the boundary conditions at zero and infinity

\[ \lim_{\chi \to 0} \{ v(\chi) F_1'(\chi) - v'(;\chi) F_1(\chi) \} = 0; \]  

\[ \lim_{\chi \to \infty} \{ u(\chi) F_1'(\chi) - u'(\chi) F_1(\chi) \} = 0. \]