COMMUTATIVE SUBALGEBRAS OF THREE FIRST-ORDER SYMMETRY OPERATORS AND SEPARATION OF VARIABLES IN THE WAVE EQUATION

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The problem of complex separation of variables in the wave equation is considered in four-dimensional Minkowskii space-time. In contrast to the known series of researches by Kalnins and Miller (see Ref. Zh., Fiz., 2B9 (1978); 1B208 and 1B209 (1979), e.g.), underlying this research is a theorem on the necessary and sufficient conditions of total separation of variables in the non-parabolic V. N. Shapovalov equation (Differents. Uravn., 16, No. 10, 1864-1874 (1980)). Nonequivalent complete sets of three differential first-order symmetry operators are constructed, appropriate coordinate systems are found, and complete separation of variables is performed in the wave equation.

All nonequivalent commutative three-dimensional subalgebras of first-order symmetry operators of the wave equation are obtained in this paper and total separation of variables is performed by using them. Let us recall that, by definition, a symmetry operator transfers every solution of a given equation into a certain solution of this same equation.

The problem of separation of variables in (1) is studied in greatest detail in [1-5] in which the separation of variables is realized within the framework of the "step-by-step" principle. Namely, if (1) allows an operator of first-order symmetry $A$, then the variable $u_1$ in (1) is separated out in the appropriate coordinate system $(u)$ where this operator can be reduced to the form $\partial/\partial u_1$, and (1) is reduced to a "shortened" equation with a smaller number of independent variables, where second-order symmetry operators are used for separation of variables or a repeated shortening is performed by using the first-order operator. The solution of (1) is represented in the form

$$q(u) = R(u) q_0(u_0) q_1(u_1) q_2(u_2) q_3(u_3)$$

in the privileged coordinate system $(u_k)$ (where the variables in (1) are separated). Here $q_k(u_k)$, $k = 0, 1, 2, 3$ are functions dependent on one variable while the function $R(u)$ can depend on several variables $u_k$. Let us note that in certain cases the "shortened" equation, in the words of the authors of [3], is complex in form and the technique they used is not effective. In our opinion, these difficulties can be associated, in particular, with the fact that the function $R$ depends on several coordinates $u_k$ and several mutually simultaneously commutative first-order symmetry operators is required to find it. The difficulties mentioned do not occur in an approach based on the Shapovalov [6] theorem about the necessary and sufficient conditions for complete separation of variables in a second order nonparabolic equation that underlies our research. According to this theorem, every privileged coordinate system is defined by a complete set of pairwise commutative linearly independent differential symmetry

*All the variables $x_i$ ($i = 0, 1, 2, 3$) are considered dimensionless. To go over to the dimensional $x_i$ the transformation $x_i = \alpha x_i$ should be carried out everywhere, where $\alpha$ has the dimensionality of a reciprocal length.


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operators of not higher than second order. Finding all the privileged coordinate systems of (1) reduces to finding all (nonequivalent) complete sets, where the necessity for a partial separation of variables and reduction of the problem to separation of variables in the "shortened" equation drops out.

Equation (1) is invariant with respect to the group K of conforal mappings of the Minkowskii space $R_{1,3}$. Generators of the group K form an algebra $k$ and the spaces $R_{1,3}$ are written in the Cartesian ($x^k$) coordinate system in the form

$$P_\alpha = \frac{\partial}{\partial x^\alpha}; \quad L_{ij} = x_i P_j - x_j P_i; \quad D = (x \cdot P) + 1; \quad K = 2 x_i (x \cdot P) - (x \cdot x) P_i + 2 x_i.$$

Here $i, j, k = 0, 1, 2, 3$; $(x \cdot x) = x_i x^i$, $x_i = g_{ik} x^k$, $(g_{ik}) = \text{diag}(+1, -1, -1, -1)$ is the metric space $R_{1,3}$ in the Cartesian ($x^k$) coordinate system. In our previous paper [8], subalgebras are obtained with the nontrivial center of the algebra $k$ as well as all nonequivalent (the definition of equivalence is given in [7]) two-dimensional commutative subalgebras of the algebra $k$ with bases $\{A_0, A_1\}$. The set $B_1, \ldots, B_r$ of elements of the algebra $k$ that commute with the elements $A_0, A_1$ form a certain subalgebra of the algebra $k$ that is also a subalgebra of one of the algebras presented in [8]. This subalgebra can be separated comparatively simply into orbits with respect to an associated representation of its groups (subgroups of the group that do not change $A_0, A_1$), whose representatives $A_{21}, A_{22}, \ldots, A_{2m}$ define the basis $\{A_0, A_1, A_{2k}\}$ of mutually nonequivalent three-dimensional commutative subalgebras of the algebra $k$. Performing an analogous operation with each of the two-dimensional commutative subalgebras listed in [8] and removing the nonequivalent triples $\{A_0, A_1, A_2\}$,

we obtain the desired nonequivalent three-dimensional commutative subalgebras with bases (4).

In conformity with (3), the elements $A_p, p = 0, 1, 2$ are scalar first-order differential operators of the form $A_p = a_p^i(x) P_i + a_p(x)$, where $a_p^i(x), a_p(x)$ are functions. From the viewpoint of the Shapovalov theorem [6], the set (4) is a complete set of the type of (3.n) of (1). Here the first digit indicates the number of first-order symmetry operators in the complete set, $n = 3$ is rank $(a_p^i a_p^q)$.

All the nonequivalent complete sets of first-order symmetry operators are represented in Table 1. The sets are divided into two groups: sets consisting of generators of Poincare subgroups of the group $K$ (1-7), and sets not reducing to one are just generators of this subgroup by any transformations from the group $K$ (8-13). The sets of the first group are distributed over the types (3.0), (3.1). Sets of the type (3.0) are divided into stationary and nonstationary. In the former case the operator $A_0 = P_0$ that results in extraction of the time variable $x^0$ in (1) and reduces it to a Helmholtz equation in three-dimensional space. The sets 1-7 result in separation of variables in the Klein-Gordon equation that was examined in [9]. There are eight nonequivalent sets of type (3.0), (3.1) for the Klein-Gordon equation. In addition to the sets 1-7 in [9], the following is still presented $A_0 = (P_0 + P_3)/2$; $A_1 = L_{01} + L_{31} + t P_2$; $A_2 = L_{20} + L_{21} + t P_2 + t P_3$; $t, \tau = \text{const}$. Since the group $K$ of the wave equation (1) is broader than the group of the Klein-Gordon equation (the Poincare equation is one such), this set turns out to be equivalent to the set 7 with respect to $K$ (see Table 1). The sets 8-13 are among the type (3.0). Because of the equivalence of the operators $L_{03}, D$ and $P_0 - K_0$, the sets 4-9 result in separation of variables in the equations [1, 3]

$$(M^2 - K^2) \varphi = -\sigma (\sigma + 2) \varphi; \quad M^2 = L_{12}^2 + L_{13}^2 + L_{23}^2; \quad K^2 = L_{31}^2 + L_{32}^2 + L_{30}^2;$$

that is obtained from the wave equation (1) by extraction of a variable associated with the operator $D$.

The sets 4-9 result in separation of variables in the equations in the Euler-Poisson-Darboux equation obtained from the wave equation (1) after extraction of the variable associated with the operator $L_{12}$. The sets 2, 8-12 result in separation of variables in the free Schrödinger equation in three-dimensional space-time, which is obtained from (1) after extraction of the variable associated with the operator $P_0 + P_3$. The sets 1 and 2 possess this same property if they are written in the equivalent form $P_0 + P_3, P_0 - P_3, P_0 + P_3, P - P_3, L_{12}$. The set 13 is missing in [10]. Therefore, still another set corresponding to this set, at least, should be appended to the 17 coordinate systems presented in [10].

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