Adiabatic invariants of charged particles are constructed that can be utilized to investigate such processes as low-frequency wave propagation in REP, amplification (absorption) of Langmuir waves in a spherically symmetric gravitational field, and particle flux interaction with a nonlinear wave in longitudinal static electrical and magnetic fields. Construction of the adiabatic invariants is executed by reduction of the original problems to a one-dimensional Hamiltonian.

A large number of exact solutions of the equations of motion of charged particles in electromagnetic fields with sufficiently high symmetry is known at this time [1]. In general, when at least certain dynamic invariants of the particles can be calculated, the situation is of considerable interest since a dynamical approach is possible in this case for the investigation of self-consistent problems in collisionless systems [2-6]. A typical problem of this kind is associated with the slow evolution of fields when adiabatic invariants exist [7, 8].

The standard method of calculating the adiabatic invariants is to construct a one-dimensional Hamiltonian formalism (HF) with Hamilton functions (FH) slowly dependent on the evolution coordinate by canonical transformations (CT) of some initial HF [9] that should result in shortening the operation because of extraction of cyclic variables. For many non-trivial problems it is sufficient to use point CT in combination with phase plane transformations.

Adiabatic invariants are constructed below for fields of certain symmetries and a circle of problems is indicated in which these invariants can be applied. The particle charge and mass and the speed of light are set equal to one in the system of units used herein.

1. CYLINDRICAL COORDINATES

The elementary operation has the following form in cylindrical coordinates

\[ dS = C\,dr + M\,d\theta + C\,dz - H\,dt, \]

where \( C = p + A \), \( p \) is the particle kinetic momentum, \( A \) is a vector potential, \( M = rC \), \( H = \gamma + \phi \), \( \gamma \) is the particle energy, and \( \phi \) is a scalar potential.

The potentials in heavy-current REP depend substantially on the coordinate \( r \) while \( \theta, z, t \) are often slow variables. For a slow dependence of the potentials on a linear combination of these variables \( L(\theta, z, t) \) we go over to a new HF governed by the equalities

\[ C_\theta = \Pi_\theta \partial L / \partial z; \quad M = \Pi_\theta \partial L / \partial \theta + \Pi_\phi; \quad H = H + \Pi_\phi \partial L / \partial t, \]

by using the point CT \( z \to L \), where \( \Pi_\theta, \Pi_\phi \) are new generalized momenta, \( H \) is a new FH and we write the shortened operation in the form

\[ dS_0 = C\,dr - (-\Pi_L)\,dL. \]

Assuming \( \Pi_L \) the Hamiltonian, we obtain the adiabatic invariant

\[ I_1 = H \, C\,dr, \]

that exists if the motion over \( r \) is finite. It is convenient to use this quantity to take...
account of slackening of the potential and investigation of low-frequency waves in REP when
the adiabaticity conditions are satisfied.

Let us note that for a perfectly magnetized REP $\theta = \theta_0$, $r = r_0$ are motion integrals
and if the field $E_z$ is quasiperiodic, then adiabatic invariants for plane fields can be
used [10]

$$I^{(1)} = \frac{\dot{\psi}}{\dot{r}} (p_z + A_z) d\psi,$$

(5)

if $\psi = f_0(t)dt - \kappa z$, i.e., for a weakly nonstationary wave

$$I^{(2)} = \frac{\dot{\psi}}{\dot{r}} (\gamma + \varsigma) d\psi -$$

(6)

for $\psi = \omega t - f_0(z)dz$ when the wave is weakly inhomogeneous.

2. SPHERICAL COORDINATES

Let us consider a particle captured by a stationary Langmuir wave being propagated in
the radial direction in the atmosphere of a centrally symmetric gravitating object. Let us
assume that the metric is defined by the Schwarzschild vacuum solution [9]

$$g^{(0)}(x) = \text{diag}\{1 - \frac{r_g}{r}, -r^2 \sin^2 \theta, -r^2, -(1 - \frac{r_g}{r})^{-1}\},$$

where $r_g$ is the gravitational radius of the object, spherical coordinates $\{x^0, x^1, x^2, x^3\} =\{t, \theta, \phi, r\}$ are utilized, and $\theta$ and $\phi$ are the polar and azimuth angles, respectively. The
tensor superscripts are in parentheses, and summation is understood to be over repeated
subscripts.

The elementary operation for a particle in a combination of gravitational and electro-
magnetic fields can be written in the form

$$d\vec{S} = P(dx^0) - Gd\tau,$$

(7)

where $\tau$ is the intrinsic time of the particle, and $P_i = -C_i = -(P_i + A_i)

$$G(P, x) = -[g^{(0)}(x) (P_{(i)} + A_{(i)}(x))(P_{(j)} + A_{(j)}(x))]^{1/2} = 1.$$

(8)

Let us note that the HF defined by (7) and (8) differs from an analogous formalism presented
in [11]. The circumstance that the formalism (7) and (8) allows only point CT is essential
since it is obtained from a variational principle under the condition $p_i p_i = 1$, and free
CT are possible only after shortening the operation (7).

In this case a stationary Langmuir wave is described by electromagnetic potential
components

$$A^{(1)} = A^{(2)} = 0; A^{(0)}, A^{(3)} = A^{(0)}, A^{(3)}(r, \phi); \dot{\psi} = \omega t + \int \kappa_3(r)\, dr,$$

(9)

$\omega \equiv \kappa_0 = \text{const}$ is the frequency measured in world time $t$, and we assume $A^{(0)}, A^{(3)}$ periodic
in $\psi$ with period $2\pi$.

Particle capture by the wave (9) is possible if

$$|\kappa_0| > \omega/(1 - r_g/r).$$

For simplicity we assume that the particle moves in a radial direction resulting in an equation analogous to (3):

$$dS_\psi = P_0 dt - (P_0 + A_{(0)}) rd\psi.$$

The point CT $t + \psi$ determines the passage to the new formalism with the evolutionary coordinate $r$, the coordinate $\psi$, the momentum $P_\psi$, and the FH $H$

$$H = \sigma (1 - r_g/r)^{-1} [(\omega P_\psi + A_{(0)}(r, \psi))^2 - (1 - r_g/r)]^{1/2} + \kappa_3(r) P_\psi + A_{(3)}(r, \phi),$$

where $\sigma = \text{sign} p_0; P_\psi = P_0/\omega = -(P_0 + A_{(0)})/\omega$.

For slow dependences of $\kappa_3$, $A_{(3)}$, $A_{(0)}$ on $r$ and $r_g \ll r$ we have the adiabatic invariant

$$I_\psi = \int_{\psi} \Pi_\psi (H, r_g/r, A_{(0)}, A_{(3)}, \kappa_3) d\psi \approx \text{const},$$

(10)