New exactly solvable models are found. The exact solutions pertain to low-lying states and are connected with the solution of the eigenvalue problem for three- and five-diagonal matrices of special form. The solution can be described by introducing an effective spin Hamiltonian that is non-Hermitian in general but has real eigenvalues. The potentials obtained go over in various limiting cases into the Mathieu, Eckart, Peschl-Teller potential and into the effective potentials for anisotropic paramagnets.

Finding and studying new exactly solvable cases of the Schrödinger equations is of interest for a great variety of physical applications. Examples are problems in quantum field theory, since the Klein-Gordon and Dirac equations in magnetic fields of certain configurations can be reduced to a Schrödinger equation [1]. Another field of applications is in solid-state physics and in the theory of molecule vibrations, where a special role is played by periodic potentials. They have also become vital recently, in connection with development of soliton theory [2], for finding periodic solutions of the Korteweg-de Vries equation that describes nonlinear dispersive media. In addition, for models such as the Kronig-Penney or Dirac-comb model [3], it is impossible to obtain simple explicit expressions for the energy levels and the wave functions, and the potential profile is rigid for all values of the parameters. This makes it impossible, for example, to describe subtle effects such as inversion splitting in ammonia [4].

Exactly solvable periodic models with flexible profiles have been observed in [5-7] (the corresponding results were used in [5] to calculate the ethylene-molecule vibration spectrum). It was shown in [6-8] that the obtained effective potentials can be used to describe paramagnets (this holds also for interacting-fermion systems of the Lipkin-Meshkov-Glick type [9]).

The present paper is a continuation of the search for new exact solutions [5-7, 10, 11] and generalizes a number of earlier results. A characteristic feature of the corresponding mode is that the exact solutions pertain only to part of the energy spectrum, since they are connected with finite-dimensional, namely spin, systems. This is not a shortcoming but a characteristic property of these models and turns out to be a basis for "generating" new exact solutions missing from [5-7, 10, 11].

The potential fields with exact solutions, considered in [6], corresponded in the simplest case to a single-axis paramagnet whose energy spectrum coincided with \( 2S + 1 \) (\( S \) is the spin) low-lying energy levels. One possible generalization was to construct effective potentials for two-axis paramagnets [7] by using the method of spin-coherent states [12]. The correspondence between the coordinate and finite-dimensional systems can be seen also from a different viewpoint. We shall attempt, without introducing spin operators from the very outset, to ascertain which generally finite-dimensional equations are suitable for the construction of effective potential. We begin with matrices having the simplest structure—tridiagonal ones.

We consider the difference equation

\[
(\omega - \varepsilon) a_{\sigma} + \rho_{\sigma} a_{\sigma+1} + \rho_{\sigma+1} a_{\sigma+1} = 0
\]

with real coefficients. By introducing a generating function (cf. [6]) we attempt to obtain a second-order differential equation of the Schrödinger type. We assume accordingly that the matrix elements that serve as coefficients in Eq. (1) are polynomials of second degree in \( \sigma \).
We choose the generating function in the form
\[ \Phi(\varphi) = \sum_{\sigma} a_{\sigma} e^{i\sigma\varphi}. \] (2)

Multiplying (1) by \( e^{i\sigma\varphi} \) and summing, we obtain a differential equation for \( \Phi \).

We shall be interested in cases in which the series in (2) terminates, so that \( \sigma \) varies in between finite limits from \( \sigma_1 \) to \( \sigma_2 \). Since we can, without loss of generality, reckon \( \sigma \) from the mean value \( \sigma_1 + \sigma_2/2 \), we assume that \( \sigma \) varies in the range \( -S \leq \sigma \leq S \), where \( S \geq 0 \) is an integer or half-integer (so that \( \sigma \) is now also integer or half-integer). For (1) to have a finite-dimensional solution, the conditions
\[ -S = p_{s+1} = 0, \] (3)
must be met. We choose accordingly the coefficients in the form
\[ a_{\sigma} = a_0 \sigma^2 + a_1 \sigma; \quad p_{\sigma} = \frac{3}{2} (\sigma + J_2) (\sigma - S - 1) ; \] (4)
\[ r_{s+1} = \frac{3'}{2} (\sigma - J_1) (\sigma + S + 1) . \]

The differential equation for \( \Phi \) then takes the form
\[ \Phi'' \left[ \alpha + \frac{(\beta + \beta')}{2} \cos \varphi + \frac{i(\beta + \beta')}{2} \sin \varphi \right] + \Phi \left[ \sin \varphi \left( (S - 1) \left( \frac{\beta + \beta'}{2} \right) - \frac{(\beta J_2 + \beta' J_1)}{2} \right) + i \alpha + i \cos \varphi \left( (S - 1) \left( \beta - \beta' \right) + \beta J_2 - \beta' J_1 \right) \right] + \Phi \left[ e + \frac{S}{2} \cos \varphi \left( \beta (J_1 + 1) + \beta' (J_1 + 1) \right) \right] = 0. \]

The condition that the coefficients of the equation be real leads to the requirements
\[ \alpha_1 = 0, \quad \beta' = \beta, \quad J_1 = J_2 = J, \]
so that the finite-difference equation (1) can be rewritten in the form
\[ (\alpha \sigma^2 - \sigma) a_{\sigma-1} + \frac{\beta}{2} (\sigma + J) (\sigma - S - 1) a_{\sigma-1} + \frac{\beta}{2} (\sigma - J) (\sigma + S + 1) a_{\sigma+1} = 0. \] (5)

We assume also that \( \alpha > \beta > 0 \) (so that the coefficient of the second derivative does not vanish).

We remove the first derivative from the differential equation, and set the coefficient of the second derivative equal to unity. This is achieved by the substitutions
\[ y = \sqrt{\frac{\alpha + \beta}{2}} \int_0^\varphi \frac{a \varphi'}{\sqrt{x + \beta} \cos \varphi'} = F\left(\frac{\varphi}{2}, \kappa\right), \quad \kappa = \sqrt{\frac{2\beta}{\alpha + \beta}}, \] (6)
\[ \cos \varphi = 2 \cosh^2 y - 1, \quad \sin \varphi = 2 \sinh y \cosh y. \]

Here \( F \) is an elliptic integral of the first kind with modulus \( \kappa \); \( \cosh, \sinh, \) and \( \cosh \) are Jacobi elliptic functions;
\[ \varphi = \text{dn} \ y', y = y'. \] (7)

As a result we obtain a Schrödinger equation
\[ \frac{d^2 \Phi}{dy^2} + \Phi (x - V) = 0, \quad x = \frac{4\varepsilon}{\alpha + \beta}. \] (8)