RELATIVISTIC THEORY OF DIRECT INTERACTIONS AND
GRAVITATIONAL TWO-BODY PROBLEM IN THE SECOND
POST-NEWTONIAN APPROXIMATION

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The three-dimensional Lagrangian of the formulation of relativistic theory of
direct interactions is applied to the problem of two-body motion in the second
post-Newtonian approximation. On the basis of the conditions of Poincare invar-
iance, taking account of the requirement of consistency with scalar-tensor theory
in the linear approximation with respect to the gravitational constant and with
the trial-particle Lagrangian in a Schwarzschild field in the approximation of the
single-body problem, the Lagrangian function of a system of two gravitating point
particles is constructed. The effectiveness of using both methods of direct-
interaction theory is shown, permitting, in particular, transition to the
Hamiltonian description of the given system.

The relativistic theory of direct particle interactions (RTDI) [1-3] is a useful alterna-
tive field approach to the problem of Poincare-invariant description of particle systems.
The four-dimensional version of this theory, based on Fokker-type action integrals, has
recently been successfully applied to gravitational problematics [2]. In the present work,
the application of three-dimensional simultaneous Lagrangian formulation of the RTDI [i,
3-5] to the problem of the motion of two gravitational bodies in the second post-Newtonian
(2PN) approximation is considered.

The first PN approximation (of order $c^{-2}$), a classical approach in general relativity
theory (GRT) [6, 7], is naturally interpreted in terms of RTDI [1, 5]. The 2PN approximation
is interesting primarily because of a series of features appearing here, which are charac-
teristic of the description of any relativistic systems of interacting particles in terms
of RTDI [4, 5, 8, 9]; for systems of gravitational particles, these features were observed
in [10-14].

The general scheme for studying the motion of a system of gravitational point bodies
[7, 10, 15] is in fact based on the possibility of considering GRT (at least for "insular
systems") as a nonlinear theory of the tensor field $g_{\mu\nu}(x)$ ($\mu, \nu = 0, 3$) against a background
of plane Minkowski space-time $M$, $\forall x$ [16]. In [10], the equations of motion of two point
bodies in the second approximation with respect to the gravitational constant $G$ were con-
structed. There is an iterative procedure leading in any order in $G$ to a Poincare-invariant
equation of motion [17]. Such equations, containing only particle variables, may be investigated by RTDI methods.

Important information on the system is included in the Lagrangian function. The general procedure proposed in [7] for constructing a variational principle corresponding to the equation of motion of a system of gravitational bodies is erroneous. It consists in the use of values of the metric \( g_{\mu\nu}(x) \) and the connectedness \( r^\lambda_{\mu\nu}(x) \) obtained by solving Einstein field equations in the field functional. However, in GRT, the equations of motion are conditions of compatibility of the field equations, and hence the solutions \( g_{\mu\nu}(x) \) and the variational principle based on them are determined not on any trajectories of the system (including virtual trajectories), but only on real trajectories satisfying the equations of motion (in the iterative approach, two orders of magnitude less than the maximum taken into account in the field equations). This variational principle does not lead to correct equations of motion [5, 18]. This feature is not apparent in the first approximation with respect to \( c^{-2} \) [7] or G [17], but in the 2PN approximation the Lagrangian constructed in [15] by this method leads to equations of motion differing by terms \( \sim c^{-4} \) from those obtained in [10] directly from the Einstein equation [11]. Confining attention here to the 2PN approximation, the Lagrangian of a system of two gravitational point bodies is constructed and investigated on the basis of the Lagrangian formalism of RTDI [1, 3-5, 9, 19]. The instantaneous form of dynamics in which the parametric equations of world lines of particles in \( M_4 \) take the form \( x^0 = ct, \ x^i = x^i_a(t) \) \((i = 1, 3; a = 1, 2)\) is used here [1, 4]. In problems of the motion of gravitational bodies, it is probably expedient to also use other forms of dynamics [3, 19], in particular, the frontal form, in which \( x^0 = ct + x^3_3(t) \), but these possibilities are not considered here.

The condition of Poincaré invariance of the Lagrangian description is expressed by a system of first-order differential equations [1-4] for the Lagrangian function

\[
L = - \sum_a m_a c^2 \gamma_a^{-1} - U, \ \gamma_a \equiv (1 - v_a/c^2)^{-1/2},
\]

where the function \( U \) describes the interaction between the particles. By the method of successive approximations in terms of \( c^{-2} \) [5], when

\[
U = \sum_{n=0}^{\infty} c^{-2n} U^{(n)},
\]

a system of equations for \( U^{(n)} \) is obtained; its general solution contains an arbitrary additive Galileo-invariant function \( \gamma_a \), which must be determined using additional conditions reflecting the physical properties of the particle system. In the given problem, these conditions are as follows: it is required, first, that the single-particle limit \((v_2 \to 0, m_2 >> m_1)\) of the function \( U^{(n)} \) coincides with the terms of corresponding order in the expansion of the accurate Lagrangian

\[
L_s = -m_1 c^2 \left\{ \frac{r - \mu}{r + \mu} - \left[ \left(1 + \frac{\mu}{r} \right)^2 + \frac{\mu}{r + \mu} \left( \frac{\mu}{r} + \frac{\mu}{r} \right) \right] \frac{\partial U}{\partial \gamma_a} \right\},
\]

describing, in harmonic coordinates, the interaction of the body with the Schwarzschild field of the massive center \( m_2; \ \mu = Gm_2/c^2 \). Second, the equivalence of the linear (in G) approximation of GRT and the theory of a scalar-tensor field is taken into account [17]. Using the general expressions obtained in [18] for the simultaneous interaction Lagrangians corresponding to a massless tensor field of arbitrary rank, the following expression is obtained for the part of the gravitational potential which is linear in G

\[
U_t = Gm_1 m_2 \sum_{\tau=0}^{\infty} \frac{(-D_0 D_3)^\tau}{c^{2\tau}(2\tau)!} \left( \left( \gamma_1 \gamma_2 \right)^{-1} - 2\gamma_1 \gamma_2 \left(1 - q_1 \cdot q_2 / c^2 \right) \right)^{\tau+1},
\]

where

\[
D_a = \sum_s s^a \delta_s \partial x_a, \ x_a = d^a x_a / dt^a \ (a = 1, 2)
\]