ADIABATIC EXPANSIONS OF GREEN'S FUNCTIONS IN ELECTROMAGNETIC AND GRAVITATIONAL FIELDS. RADIATIVE CORRECTIONS

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A solution of Maxwell's equations is constructed that describes a homogeneous electromagnetic field in a curved space-time in the lowest adiabatic approximation in the curvature. The Green's function for Dirac particles in this field is constructed to the same accuracy. It is shown that in the case of a super strong magnetic field the main approximation is the two-dimensional approximation, moreover, the gravitational correction is determined by a single component of the curvature tensor $R_{1230}$. The mass operator of the electron is calculated in this approximation.

1. Homogeneous Electromagnetic Field

We write down the equation for the potential of a locally homogeneous constant electromagnetic field in curved space-time

$$\nabla^2 \tilde{A}_\nu - R_{\mu}^{\nu} A_\mu = 0. \quad (1)$$

Using the expansions for the metric tensor and the Christoffel symbols in the normal Riemann coordinates \([1, 2]\)

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\nu\gamma\delta} y^\gamma y^\delta, \quad (2)$$

$$\Gamma^\gamma_{\mu\nu} = - \frac{1}{3} [R^\gamma_{\mu\nu} + R^\gamma_{\nu\mu}] y^\nu, \quad (3)$$

we write (1) in the form

$$\partial^\mu \partial_\mu A_\alpha + \frac{1}{3} R^\gamma_{\mu\delta} y^\nu y^\eta \partial_\nu \partial_\eta A_\mu - \frac{2}{3} R^\mu_{\alpha\mu} A_\mu - \frac{2}{3} R^\mu_{\nu\delta} y^\nu \partial_\delta A_\mu + \frac{2}{3} [R^\mu_{\nu\delta} + R^\mu_{\delta\nu}] y^\nu \partial_\delta A_\mu = 0. \quad (4)$$

Here $\nabla_\alpha$ is the covariant derivative, $\partial_\alpha = \partial / \partial y^\alpha$, $y^\alpha = (x - z)^\alpha$, where $z$ is the origin of the normal Riemann coordinate system in which the curvature tensor is calculated

$$R^\alpha_{\nu\rho\tau} = R^\alpha_{\nu\rho\tau}(z).$$

The potential $A_\mu$ can be expressed in the form of an adiabatic expansion

$$A_\mu = A^{(0)}_\mu + A^{(1)}_\mu + A^{(2)}_\mu + \ldots, \quad (5)$$

where $A^{(k)}_\mu$ contains the geometric part that corresponds to all the quantities in the $k$-th derivative of the metric tensor. Clearly $A^{(1)}_\mu \equiv 0$, i.e., the corresponding term in the expansion of the form (2) is absent. The substitution of (5) into (4) leads to a system of coupled equations,

$$\partial^\mu \partial_\mu A^{(0)}_\mu = 0, \quad (6)$$

$$\partial^\mu \partial_\mu A^{(2)}_\mu = \frac{2}{3} R^\mu_{\alpha\mu} A^{(0)}_\alpha + \frac{2}{3} R^\mu_{\nu\delta} y^\nu \partial_\delta A^{(0)}_\mu - \frac{1}{3} R^\gamma_{\mu\delta} y^\nu y^\eta \partial_\nu \partial_\eta A^{(0)}_\mu - \frac{2}{3} [R^\mu_{\nu\delta} + R^\mu_{\delta\nu}] y^\nu \partial_\delta A^{(0)}_\mu, \quad (7)$$

moreover, in the zeroth approximation we have for the constant homogeneous electromagnetic field

$$A^{(0)}_\mu = - \frac{1}{2} \tilde{F}_{\mu\nu} y^\nu. \quad (8)$$

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where $F_{\mu\nu} = \text{const}$. This means that the tensor $F_{\mu\nu}$ is a "true" field tensor at the origin of the normal Riemann coordinate system. Taking into account (8), we find from $A_\mu^{(2)}$ Eq. (7), which added to $A_\mu^{(0)}$ should satisfy the covariant Lorenz condition

$$\nabla_{\mu}A^{\mu} = 0.$$  

As a result of some calculations we obtain [3]

$$A_\mu^{(2)} = D_{\mu\nu}y^\nu y^\nu,$$  

where

$$D_{\mu\nu}y^\nu = -\frac{1}{16} R^\alpha_{\mu\nu} F_{\alpha\beta}y^\beta + \frac{5}{288} R^\rho_{\mu\nu} F_{\alpha\beta}y^\beta + \frac{1}{24} R^\alpha_{\mu\nu} F_{\rho\xi}y^\rho y^\xi - \frac{11}{288} R^\gamma_{\mu\nu} F_{\alpha\beta}y^\alpha y^\beta.$$  

2. Dirac Equation

We write down the generally covariant Dirac equation for the Green's function in an external field

$$(i\gamma^\mu(x) - m) G(x, x') = \delta(x - x').$$  

Here the matrices $\gamma^\mu(x)$ satisfy the relations

$$\gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = 2\delta^{\mu\nu}(x),$$  

where $\gamma^\mu = \partial^\mu - \Gamma^\mu + i\alpha^\mu$ is the "extended" spinor covariant derivative, and $\Gamma^\mu$ is the spinor connection. Introducing a normal Riemann coordinate system with the origin at the point with the coordinates $z_1$, and defining $y = (x - z)$, $y' = (x' - z)$ and using the adiabatic expansions for $\Gamma^\mu$ and $\gamma^\mu(x)$ [1, 2], by taking into account (8) and (10) we write (12) in the form

$$(i\gamma^\mu F_{\mu\nu}y^\nu - m) G(x, y') = \delta(y - y').$$  

By separating the exponential factor [4]

$$G(y, y') = \exp \left[\frac{ie}{2} F_{\mu\nu}y^\mu y^\nu\right] S(y, y'),$$  

we will look for a $S(y, y')$ in the form of an adiabatic expansion

$$S = S_0 + S_1 + S_2 + ..., $$  

for which $S_1 = 0$, and $S_0$ and $S_2$ satisfy the equations

$$\left\{i\gamma^\mu\partial_\mu + \frac{e}{2} \gamma^\nu F_{\nu\lambda}u^\lambda - m\right\} S_0 = \delta(u),$$  

$$\left\{i\gamma^\mu\partial_\mu + \frac{e}{2} \gamma^\nu F_{\nu\lambda}u^\lambda - m\right\} S_2 = F(u, y') S_0,$$  

$$F(u, y') = \left\{\frac{i}{8} \gamma^\mu \gamma^\nu R^\rho_{\mu\nu} [u + y']^\rho - \frac{i}{6} \gamma^\nu R^\rho_{\mu\nu} \times \right\}$$

$$\times [u + y']^\rho [u + y']^\xi \partial_\mu - \frac{e}{12} \gamma^\nu R^\rho_{\mu\nu} F_{\rho\xi} [u + y']^\rho [u + y']^\xi \times \right\}$$

$$\times u^\mu + e \gamma^\nu D_{\nu\rho\lambda} [u + y']^\rho [u + y']^\xi [u + y']^\lambda \right\},$$

where $u = y - y'$. Transforming to the momentum representation with respect to the $u$ variables

$$S_1 = \int \frac{d\kappa}{(2\pi)^4} e^{-ie\kappa u} S_1(\kappa, y'),$$  

we obtain (the Fourier transform of $S_0$ does not depend on $y'$):

$$\left\{\gamma^\mu \kappa_\mu - \frac{ie}{2} F_{\nu\rho} \frac{\partial}{\partial \kappa_\nu} - m\right\} S_0(\kappa) = 1,$$  

$$\left\{\gamma^\mu \kappa_\mu - \frac{ie}{2} F_{\nu\rho} \frac{\partial}{\partial \kappa_\nu} - m\right\} S_1(\kappa, y') = F(\kappa, y') S_0.$$  

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