d'Alambert equation lies in the fact that, as it has been shown above, spaces of their non-trivial fixed-order symmetries are reducible.

LITERATURE CITED


METHODS OF GENERATING INTEGRABLE POTENTIALS FOR THE SCHRODINGER EQUATION AND NONLOCAL SYMMETRIES

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Methods of generating exactly integrable potentials for the Schrödinger equation are consolidated within the framework of a simple construction. The Abraham-Moses method is generalized to the case of the nonstationary Schrödinger equation. An algorithm is proposed for solving the Schrödinger equation based on nonlocal symmetry operators.

In [1, 2] a method is proposed, and described in a simplified form, for generating potentials that yield exact solutions of the nonstationary Schrödinger equation.*

In this paper the various generation methods [1-8] are consolidated within the framework of a simple construction. An algorithm is proposed to solve the Schrödinger equation based on nonlocal symmetry operators and including the generation procedure. We obtain new exactly integrable potentials as well as explicit formulas for the corresponding nonlocal symmetry operators for the Schrödinger equation. We note that nonlocal symmetries raise expectations [9-13] of expanding the class of exactly integrable equations, because local symmetries associated with the Lie-Backlund group lead to a rather narrow, albeit important, class of integrable equations, studied in detail in the literature [13, 14].

1. GENERATION METHODS

The generation of potentials for the Schrödinger equation

\[ \tilde{H}\psi(x) \equiv (-d_{xx}+u(x))\psi(x) = \varepsilon \psi(x) \tag{1} \]

consists in transforming a given solution \( \psi(x) \) with the help of an operator

\[ \psi(x) = S \psi(x) \tag{2} \]

in such a way that the function \( \psi(x) \) satisfies Eq. (1) with a new potential \( u(x) \):

\[ \tilde{H}\psi(x) \equiv (-d_{xx}+u(x))\psi(x) = \varepsilon \psi(x). \]

Here \( d_x = d/dx; \quad d_{xx} = d_x d_x; \) and \( \varepsilon \) is a parameter. The formulas relating the potentials \( a(x) \) and \( u(x) \) to each other depend on the specific form of the operator \( S \). It is sufficient to require the existence of the left inverse for \( S \).

*By exact integrability we understand construction of the full system of solutions.

The differences between various generation methods lie in the choice of \( S \); \( S \) can be local (differential) or nonlocal (integral or integrodifferential). We shall distinguish between local and nonlocal generation methods. The Darboux method [3, 7, 8] belongs to the first group while the Abraham-Moses transformation and its modifications [4-6] belong to the second one.

In the classical Darboux method, the transformation (2) is effected by the operator

\[ S = \partial_x + b(x), \]

where \( b(x) \) is determined through \( \psi(x) \).

In a series of papers [15, 16], the Darboux method has been modified for matrix equations. In [8], a certain generalization of the Darboux method was given for the three-dimensional Schrödinger equation. In this case, \( H = -\Delta + a(r) \) and the operators \( S \) should be chosen in the form \( S = p(r) + (q(r) \cdot \nabla) \). Here \( p(r) \) and \( q(r) \) are functions; \( \Delta \) is the Laplace operator; and \( \nabla \) is the gradient operator.

Local generation methods are easy to generalize. Indeed, for the nonstationary Schrödinger equation with the potential \( a(r, t) \)

\[ F\psi(r, t) = (i\partial_t + \Delta - a(r, t)) \psi(r, t) = 0 \] (3)

we determine the operator \( S \) by the transformation (2)

\[ \psi(r, t) = S\psi(r, t) \] (4)

from the condition

\[ [S, F] = SF - FS = w(r, t)S + PF, \] (5)

where \( w(r, t) \) is a real function of coordinates and time and \( P \) is a linear operator that plays the role of a Lagrange multiplier.

**LEMMA.** The function \( \psi(r, t) \) of the form (4) satisfies the Schrödinger equation

\[ F\psi(r, t) = (i\partial_t + \Delta - u(r, t)) \psi(r, t) = 0 \] (6)

with the potential

\[ u(r, t) = a(r, t) - w(r, t). \] (7)

The classical Darboux method can be trivially generalized to the case of the nonstationary one-dimensional Schrödinger equation. In (4) and (5) we take

\[ F\psi(x, t) = (i\partial_t + \partial_x - a(x, t)) \psi(x, t) = 0; \]

\[ S = \partial_x + b(x, t), \ P = 0. \] (8)

From (5) it follows directly that

\[ b = -\partial_x \ln|\Phi(x, t)|, \]

where \( \Phi(x, t) \) is a particular solution of Eq. (8), i.e.,

\[ (i\partial_t + \partial_x - a(x, t))\Phi(x, t) = 0. \]

The transformed potential (7) has the form

\[ u(x, t) = a(x, t) - 2\partial_x \ln|\Phi(x, t)|. \]

For a stationary potential \( a(x, t) = a(x) \) and a stationary solution \( \Phi(x, t) = \exp(i\varepsilon t)\Phi(x) \), we obtain the formulas of the classical Darboux method.

Further generalizations can be obtained by choosing as \( S \) a differential operator of order higher than one for one or several variables. However, in contrast with the classical Darboux method, it is not possible here to solve the system (5) in the general case for an arbitrary initial potential in Eq. (3) and to find an expression analogous to (9).