NONCOMMUTATIVE INTEGRATION OF THE DIRAC EQUATION IN RIEMANN
SPACES POSsessING A GROUP OF AUTOMORPHISMS

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Applying the method of noncommutative integration for linear differential equations, we build exact solutions for the Dirac equation in 4-dimensional Riemann spaces, which have a 5-parameter group of automorphisms and where the Klein-Gordon and the Dirac equations are nonintegrable using the technique of complete separation of variables.

The present paper is a direct continuation of [1], where the method of noncommutative integration for linear differential equations in partial derivatives [2] was used to integrate the Klein-Gordon and Dirac equations in four-dimensional Riemann spaces that have a group of automorphisms and does not support integration using the complete separation of variable technique. From this point of view, there are four metrics of interest that admit a 5-parameter group of automorphisms (see [3, p. 312]). In [1], the Klein-Gordon equation has been integrated in explicit form for all four metrics, and the noncommutative integration technique for the Dirac equation has been demonstrated using one of them as an example.

In the present paper, we present in explicit form the results of noncommutative integration of the Dirac equation for the remaining metrics. We adopt the definitions and notations of [1].

We write the Dirac equation in the Riemann space $V_4$ with the metric tensor $g_{ij}$ in the form

$$H \psi (x) \equiv \left( i \gamma^i \left( \partial_x - \frac{1}{4} \gamma^j \gamma^j \right) - m \right) \psi (x) = 0. \quad (1)$$

Here, $i, j, k, \ell = 1, 2, 3, 4$; $\partial_x^k = \partial / \partial x^k$. The Dirac matrices $\gamma^i (x)$ in the space $V_4$, $\gamma^i \gamma^j + \gamma^j \gamma^i = 2g^{ij}(x)$, are expressed through the constant Dirac matrices $\gamma^a (a, b, c = 1, 2, 3, 4; \gamma^a \gamma^b + \gamma^b \gamma^a = 2\delta^{ab})$, with the help of the orthogonal tetrad $e^a_i(x)$, $\gamma^j (x) = e^a_a(x) \gamma^a$; $\gamma^i_j$ is the covariant derivative of $\gamma^i$ over $x^j$ relative to the metric $g_{ij}$.

Let the metric $g_{ij}$ of the space $V_4$ admit a group of automorphisms, which is determined by the Killing vectors $Y^a_x = \xi^a(x) \delta_j$, which for the metrics under consideration form a solvable Lie algebra with commutation relations given by

$$\{ \xi^a, \xi^b \} = \lambda^{ab}_c \xi^c ,$$

where $\lambda^{ab}_c$ are the structure constants of the Lie algebra.

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Then the operators

\[ X_a = \psi^i \left( \partial_i - \frac{1}{4} \gamma_j \gamma_i \right) - \frac{1}{8} [\gamma_i, \gamma^a] \xi_{a;\iota} \equiv Y_a + x_a \]  

(2)

commute with the operator \( H \) of the Dirac equation (1) and realize the same Lie algebra, i.e., \([X_a, X_B] = c_{\alpha\beta} \gamma_X Y \).

Therefore, as in [1], we use the same \( \lambda \)-representations (the operators \( X_a, \xi \), \( [X_a, \xi_B] = -c_{\alpha\beta} \gamma_X Y \)) for each of the metrics under consideration.

The basis for the space of solutions of the Dirac equation (1) is derived using the method of noncommutative integration by means of solving the following system of the equations

\[ X_a \psi = \lambda \psi; \]
\[ H \psi = 0. \]  

(3)  
(4)

Below, we give the results of the noncommutative integration for the Dirac equation (1) in the spaces \( \mathcal{V}_a \) with the metrics (10), (15), and (21) of [1]. To write the metric \( g_{ij} \), we use the linear element \( \mathcal{e}^{ij} = g_{ij} \mathcal{d}x^i \mathcal{d}x^j \).

1.

\[ ds^2 = 2K_{ij} \exp \left( -\frac{1}{2} x^i \right) \mathcal{d}x^i \mathcal{d}x^j + K_{23} \exp (-x^3)(\mathcal{d}x^1 + x^i \mathcal{d}x^i)^2 + K_{23} \exp (-2x^1)(\mathcal{d}x^2)^2 - \left( \frac{1}{4} K_{11}/K_{23} \right)(\mathcal{d}x^1)^2. \]  

(5)

Here and below \( K_{ij} = \text{const} \).

The group of automorphisms, which this metric admits, is determined by five Killing vectors

\[ Y_1 = \partial_1; \quad Y_2 = \partial_2; \quad Y_3 = \partial_3 + x^i \partial_i; \quad Y_4 = -\frac{1}{2} x^i \partial_i + \frac{1}{2} x^i \partial_i + x^i \partial_i + \partial_i; \]
\[ Y_5 = (-x^3 - x^1 x^2 + q \exp \left( \frac{1}{2} x^1 \right)) \partial_1 + x^3 x^1 \partial_2 + (x^1)^2 \partial_3 + 2x^1 \partial_4; \]
\[ q = K_{13}/K_{23}. \]  

(6)

The \( \lambda \)-representation is given by the operators

\[ l_1 = \lambda; \quad l_2 = \partial/\partial \mu; \quad l_3 = -\mu \lambda; \quad l_4 = \frac{1}{2} \lambda \partial/\partial \mu - \mu \partial/\partial \mu; \]
\[ l_5 = -\mu \partial/\partial \lambda + \mu \partial/\partial \mu - \lambda/\mu^2. \]  

(7)

Here \( \lambda, \mu \), and \( I \) are the parameters (in general complex) labeling the basis for the space of solutions of Eq. (1).

The tetrad \( e_{a}^{i} \) is given by

\[ (e_1^i) = (-i\sigma/K_{13}, 0, i \exp(x^1/2)/\sigma, 0); \]
\[ (e_2^i) = (-x^1 \sqrt{K_{23}/K_{13}}, \exp(x^1/2)/\sqrt{K_{23}}, 0, 0); \]
\[ (e_3^i) = (0, 0, \exp(x^1/2)/\sigma, 0); \]
\[ (e_4^i) = (0, 0, 0, 2 \sqrt{K_{23}/K_{13}}); \]
\[ \sigma^2 \equiv K_{33} \exp (-x^1) + K_{33} (x^1)^2. \]