ON THE ELIMINATION OF THE CRITICAL TERM IN A
GENERAL FIRST-ORDER URANUS–NEPTUNE THEORY

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Abstract. A solution of the Uranus–Neptune planetary canonical equations of motion through the Von Zeipel technique is presented. A unique determining function which depends upon mixed canonical variables, reduces the 1:2 critical terms of the Hamiltonian to the set of its secular terms. The Poincaré canonical variables are used. We refer to a common fixed plane, and apply the Jacobi–Radau set of origins. In our expansion we neglected terms of power higher than the fourth with respect to the eccentricities and sines of the inclinations.

1. Introduction

The introduction of the determining function involved in the Von Zeipel’s procedure was first attempted by Von Zeipel in his researches, dealing with the construction of a qualitative theory of minor planets, under the influence of attraction of Sun and Jupiter in order to eliminate short and long periodic terms. In his masterful study, he applied the principles of H. Poincaré, Delaunay, and E. W. Brown with certain necessary modifications.

The determining, the generating, or the characteristic function could be expressed in a series form, and assumed to be obtained pragmatically. Beyond doubt are the existence, uniqueness, and convergence properties of this function. Elimination of one term at a time from the Hamiltonian of the canonical set of equations of motion was attempted by Delaunay in his Lunar theory. Brouwer thought that by applying the Von Zeipel’s method, we can acquire a Lunar theory more efficient than that of Delaunay in some respects. Jean Meffroy explored its application in his general planetary theory in the case of two planets.

2. Canonical Equations of Motion

The twelve planetary canonical equations of motion for planets 1 and 2, where index 1 and 2 refer to planets Uranus and Neptune respectively, in terms of the Poincaré canonical variables can be stated as follows:

$$\frac{dL_u}{dt} = \frac{\partial F}{\partial \lambda_u}, \quad \frac{d\lambda_u}{dt} = -\frac{\partial F}{\partial L_u},$$

$$\frac{dH_u}{dt} = \frac{\partial F}{\partial K_u}, \quad \frac{dK_u}{dt} = -\frac{\partial F}{\partial H_u}, \quad (u = 1, 2)$$

(1)
\[
\frac{dP_u}{dt} = \frac{\partial F}{\partial Q_u}, \quad \frac{dQ_u}{dt} = -\frac{\partial F}{\partial P_u};
\]

where

\[
L_u = \beta_u \sqrt{\left( k^2 m_0 + m_1 + \ldots + m_{u-1} a_u \right)},
\]

\[
\lambda_u = l_u + g_u + h_u,
\]

\[
H_u = \sqrt{2(L_u - G_u)} \cos (g_u + h_u),
\]

\[
K_u = -\sqrt{2(L_u - G_u)} \sin (g_u + h_u),
\]

\[
P_u = \sqrt{2(G_u - H_u)} \cos h_u,
\]

\[
Q_u = -\sqrt{2(G_u - H_u)} \sin h_u;
\]

\[L_u, \lambda_u, H_u, K_u, P_u, Q_u\] are the Poincaré canonical variables, and \[L_u, l_u, G_u, g_u, H_u, h_u\] are the Delaunay variables. Moreover, while

\[
F \sim F_0 + F_1 = \frac{k^2 m_0 \beta_1}{2a_1} + \frac{k^2 m_0 \beta_2}{2a_2} + k^2 m_0 \beta_2 \left( \frac{1}{r_{02}^2} - \frac{1}{r_2^2} \right) + \frac{\sigma k^2 \beta_1 \beta_2}{r_{12}};
\]

i.e., \(F\) is restricted to the zero- and first-order Hamiltonian with respect to a parameter \(\sigma\), which is of the order of the planetary masses. We adopt the following notations:

- \(m_0\) : mass of the Sun;
- \(k^2\) : constant of gravitation;
- \(\beta_1, \beta_2\) : finite numerical coefficients;
- \(\sigma\) : small parameter of the order of planetary masses:
- \(r_{01}, r_{02}\) : distances of planets Uranus and Neptune from the Sun;
- \(r_2\) : distance of Neptune from the center of mass of Sun and Uranus;
- \(r_{12}\) : mutual distance between Uranus and Neptune;
- \(a_1, a_2\) : semi-major axes of osculating ellipses of Uranus and Neptune;
- \(\alpha \beta_1\) : mass of Uranus;
- \(\alpha \beta_2\) : mass of Neptune;
- \(\theta_{12}\) : angle between vectors \(r_1\) and \(r_2\).

The first order Hamiltonian reduced to the sum of terms of degree 0, 1, with respect to \(\sigma\) may be written as

\[
F \sim F_0 + F_1 = \frac{k^2 m_0 \beta_1}{2a_1} + \frac{k^2 m_0 \beta_2}{2a_2} - \frac{\sigma k^2 \beta_1 \beta_2 r_{01}}{r_2^2} \cos \theta_{12} + \frac{\sigma k^2 \beta_1 \beta_2}{\Delta_{12}};
\]

\(\Delta_{12}\) being defined by the equality

\[
\Delta_{12}^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos \theta_{12}.
\]