AN APPROACH TO THE RADIAL DISTRIBUTION OF PLANETARY ORBITS

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Abstract. The problem of determination of the radial distribution of the planetary orbits is approached under the assumption that the average present radial sizes of the orbits were already determined when the protoplanetary cloud flattened by initial angular momentum aggregated into a set of concentric rings from which the planetary material was ultimately collected. The object of this argument is to derive a consistent stationary distribution of orbits so that the problem of the non-stationary formation of the orbital rings is not here considered. Under the flattening assumption the 3D Poisson equation is replaced by the 2D Helmholtz equation (inhomogeneous) which is solved by use of an averaging theorem generalization of the well-known averaging theorem for the homogeneous Helmholtz equation. Augmenting the ring potentials obtained by specializing the mass distribution in the disk by a solar potential term and a rotational potential, differentiation leads to a generalization of the Kepler 3D Law suitable for the many-body problem of a solar system with circular orbits. In this way a system of transcendental equations involving Bessel functions of the first and second kind are obtained which must be satisfied by the orbital radii. Naturally the restriction to circular orbits represents only an approximation to the orbital determination problem, but considering that no arguments have previously been available for the determination even of circular orbits it would seem to represent an advance.

In the present note a gravitational potential is derived for the Sun and a set of concentric circular rings about it in terms of the Bessel functions of the first and second kind (cf. Watson, 1944). The method used depends upon an averaging theorem for the inhomogeneous Helmholtz equation which is obtained as follows.

Starting from a gravitational potential \( u \) which satisfies Poisson’s equation

\[
\Delta u = -4\pi G \rho,
\]

with \( G \) the gravitation constant and \( \rho \) the mass per unit volume we assume that the protoplanetary cloud (see Safronov, 1972) has already been flattened to a disk shape in consequence of initial angular momentum and this can be described by the following assumptions

\[
u = e^{-k|z|} \nu(r, \theta),
\]

\[
\rho = e^{-k|z|} k \sigma / 2,
\]

\[
\sigma = \int_{-\infty}^{\infty} \rho \, dz;
\]

where \( \sigma \) is the mass per unit disk area. The disc thickness is roughly \( 1/k \). Thus \( \nu \) is seen to satisfy the inhomogenous Helmholtz equation.
\[ \Delta v + k^2 v = -2\pi k G \sigma. \]  

(5)

Since only the radial dependence is of interest here it is convenient to use the following averaging theorem which will be proved below: namely,

\[ \langle v \rangle_L = v_0 J_0(k\alpha) - \frac{\pi k G}{2} \int_0^\alpha \begin{vmatrix} J_0(kr) & J_0(ka) \\ Y_0(kr) & Y_0(ka) \end{vmatrix} d\alpha, \]  

(6)

where \( \langle v \rangle_L \) denotes the average value of \( v \) on the circumference of a circle of radius \( \alpha \) and \( v_0 \) its value at the center while the surface integral is extended over the interior of the circle \( r \leq \alpha \) and \( J_0 \) is the Bessel function of the first kind of zero order and \( Y_0 \) is the (Weber) Bessel function of the second kind of zero order defined on p. 64 of Watson (1944), Equation (6) may be established as follows. In Green's theorem

\[ \int (v \Delta w - w \Delta v) d\alpha = \int (v \partial_n w - w \partial_n v) d\alpha \]  

(7)

let \( \Delta v + k^2 v = -2\pi k G \sigma \), so that the left-hand side of Equation (7) may also be written as

\[ \int v(\Delta w + k^2 w) - w(\Delta v + k^2 v) d\alpha. \]

Taking \( w_1 = \frac{1}{2} \pi a J_0(kr) \) one has

\[ \Delta w_1 + k^2 w_1 = 0, \]  

(8)

\[ \partial_r w_1 = \partial_n w_1 = \frac{1}{2} \pi k a J'_0(kr); \]  

(9)

and integrating Equation (7) over \( r \leq \alpha \) we obtain

\[ \frac{\pi a}{2} 2\pi k G \int \alpha J_0(kr) d\alpha = \frac{\pi a}{2} k J'_0(ka) \hat{\delta} v d\alpha - \frac{\pi a}{2} J_0(ka) \hat{\delta} \partial_n v d\alpha. \]  

(10)

Taking \( w_2 = \frac{1}{2} \pi a Y_0(kr) \) we have

\[ \Delta w_2 + k^2 w_2 = 2\pi a \delta(r), \]  

(11)

\[ \partial_r w_2 = \partial_n w_2 = \frac{1}{2} \pi k a Y'_0(kr), \]  

(12)

\[ \int \delta(r) d\alpha = 1. \]  

(13)

Accordingly integrating Equation (7) over \( r \leq \alpha \) we obtain

\[ 2\pi a v_0 + \frac{\pi a}{2} 2\pi k G \int \alpha Y_0(kr) d\alpha = \frac{\pi a}{2} k Y'_0(ka) \hat{\delta} v d\alpha - \frac{\pi a}{2} Y_0(ka) \hat{\delta} \partial_n v d\alpha. \]  

(14)