SOLUTION OF THE DIRAC EQUATION IN QUATERNIONS

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A technique is considered for solving in quaternions the Dirac equation for particles with an anomalous magnetic moment in the field of a plane electromagnetic wave.

A minimum algebraic realization of the Dirac equation was proposed in [1], in the sense that the number of independent components of the wave function equals the rank of the algebra. The obtained equations, in contrast to those previously known [2, 3], are homogeneous over the field of complex numbers:

\[ \bar{\nabla} \Psi(x) \Pi_1 + \bar{\nabla} \Psi(x) \Pi_2 = m \Psi(x) S_1. \] (1)

A representation of the Lorentz group of the following form is realized on the wave function \( \Psi(x) \), which is a complex quaternion:

\[ \Psi^L(x) = L^* \Psi(x^L) \Pi_1 + L \Psi(x^L) \Pi_2, \] (2)

where \( x^L = Lx \overline{L} \), \( L = \frac{1 + q}{\sqrt{1 - q q}} \), \( \overline{L} = 1 \) is a complex quaternion parameter, \( q \) are constant quaternions with the properties

\[ \Pi_i \Pi_j = \delta_{ij} \Pi_j, \quad \Pi_1 + \Pi_2 = 1, \quad S_i \Pi_{ij} = \Pi_{ij} S_i, \quad S_i S_j + S_j S_i = 2 \delta_{ij}, \] (3)

The most general expression for these quaternions follows from (3):

\[ \Pi_1 = \frac{1}{2} (1 + i n_3), \quad \Pi_2 = \frac{1}{2} (1 - i n_3), \quad S_1 = i n_1, \quad S_2 = i n_2. \]

The real quaternions \( n_1, n_2, n_3 \) form the basis of the quaternion algebra \( n_i n_j = \epsilon_{ijk} n_k - \epsilon_{ijk} \).

Transforming to the momentum representation \( \Psi(x) = \int \tilde{\Psi}(p) e^{i(p \cdot x + m t)} dp \), we obtain in place of (1) the algebraic equation:

\[ p \bar{\nabla} \tilde{\Psi}(p) \Pi_1 + \bar{\nabla} \tilde{\Psi}(p) \Pi_2 = -i m \tilde{\Psi}(p) S_1. \] (4)

Projections of (4) onto \( \Pi_1 \) and \( \Pi_2 \) give a system compatible for \( pp = -m^2 \). The general solution of (4) is then

\[ \tilde{\Psi}(p) = u - l \frac{p}{m} u S_1, \] (5)

*On the complex quaternions of the form \( z = z_0 + z' \), \( \bar{z} = \bar{z}_0 + \bar{z}' \) are defined in the operations:

\[ z^a = z_0^a + z'_0, \quad \bar{z}^a = \bar{z}_0^a + \bar{z}'_0, \quad z z' = z_0 \bar{z}_0' - z_0' \bar{z}_0 + z_0 z' + \bar{z}_0' \bar{z}_0 + [z z']. \]
where the condition \( u \Pi_2 = u \) is imposed on the quaternion \( u \).

In the Hamiltonian form Eq. (1) has the form

\[
-i \frac{\partial \Psi}{\partial t} = \Psi (\Pi_2 - \Pi_1) + m \Psi S_z.
\]

(6)

This equation is clearly invariant with respect to three-dimensional rotations, when \( q^\alpha = q = e \). The transformation law (2) gives an expression for the infinitesimal generators which commute with the Hamiltonian \( H = \Psi (\ldots) (\Pi_2 - \Pi_1) + m (\ldots) S_z \) from (6):

\[
I (e) \Psi = i \left( \frac{c + (xc - cx) \Psi + [(xc - cx) \Psi]}{4} \right) \Pi.
\]

(7)

An additional classification of solutions (5) can be carried out with the help of the spin projection operator onto the momentum \( \sigma = i \left( \frac{p}{|p|} \right) = \frac{i}{2} \frac{p}{|p|} \), whose eigenfunctions are \( \Psi_\pm (\rho) = \left( 1 \pm \frac{p}{|p|} \right) u \).

The equation for a spinor particle interacting with an external electromagnetic field \( A \) is written in the formalism under consideration in the following form:

\[
(\vec{\gamma} - ieA) \Psi \Pi_1 + (\bar{\gamma} - ie\bar{A}) \Psi \Pi_2 + \beta F \Psi \Pi_3 S_1 + \beta F^* \Psi \Pi_3 S_1 = m \Psi S_z,
\]

(8)

with consideration of anomalous magnetic and dipole electric moments of the particle. The complex numbers \( \alpha \) and \( \beta \) can be limited by the requirement that (8) be invariant with respect to \( P, C \), and \( T \) transformations. Let us consider the case of a plane electromagnetic wave \( A = A(\xi) (\vec{\xi} = \kappa \vec{x} + \bar{\kappa} \vec{x}, \kappa \bar{\kappa} = 0) \), which has been studied, for example, in [4] on the basis of the matrix formulation of Eq. (8). Then

\[
F = - \bar{\gamma} A = - \bar{\kappa} \frac{dA}{d\bar{\xi}}.
\]

(9)

From the Lorentz condition \( \vec{\gamma} A + A \bar{\gamma} = 0 \) follows the relations

\[
- \frac{\kappa}{\bar{\kappa}} \frac{dA}{d\bar{\xi}} = \frac{d\bar{A}}{d\bar{\xi}}, \quad \frac{\kappa}{\bar{\kappa}} \frac{d\bar{A}}{d\bar{\xi}} = 0.
\]

We will seek the solution (8) in the form

\[
\Psi (\xi) = \Psi (\bar{\xi}) e^{(\rho \bar{\xi} + \bar{\rho} \xi)},
\]

(10)

where \( \rho \bar{\rho} = - m^2 \). Then

\[
\left( \frac{\kappa}{\bar{\kappa}} \frac{d}{d\bar{\xi}} + P \right) \Psi \Pi_1 + (\beta F^* - m) \Psi \Pi_2 S_1 = 0;
\]

\[
\left( \frac{-\bar{\kappa}}{\kappa} \frac{d}{d\xi} + \bar{P} \right) \times \Psi \Pi_2 + (\bar{\alpha} F - m) \Psi \Pi_1 S_1 = 0,
\]

(11)

where \( P = i (\rho - eA) \). Multiplying the first equation of system (10) by \( \bar{\kappa} \), and the second by \( \kappa \), we obtain the system of algebraic equations

\[
\bar{\kappa} \rho \Psi \Pi_1 = \bar{\kappa} m \Psi \Pi_2 S_1, \quad \kappa \bar{\rho} \Psi \Pi_2 = \kappa m \Psi \Pi_1 S_1.
\]

(12)

connecting the quaternions \( \Psi \Pi_1 \) and \( \Psi \Pi_2 \). The general solution of (12) is

\[
\Psi = (\bar{\rho} \kappa + m \bar{\kappa}) \Phi \Pi_1 + (\rho \bar{\kappa} + m \kappa) \Phi \Pi_2 S_1,
\]

(13)

where \( \Phi \) is an arbitrary quaternion. Substitution of (13) into (11) gives

\[
\left( \kappa \bar{\rho} \frac{d}{d\bar{\xi}} + iD \right) \kappa \bar{\Phi} \Pi_1 + \beta F^* \bar{p} \bar{\kappa} \Phi \Pi_1 = 0; \quad \left( \bar{\kappa} p \frac{d}{d\xi} + iD \right) \bar{\kappa} \Phi \Pi_1 + \bar{\alpha} \bar{F} \bar{p} \bar{\kappa} \Phi \Pi_1 = 0,
\]

(14)