Schiff's effect is examined within the framework of the monadic method for defining the reference frame for the motion of a rotating body along ballistic trajectories in a Schwarzschild field. A general equation is obtained for calculating the precession angle in a comoving coordinate system. Examples of the motion of a gyroscope along parabolic, elliptic, and hyperbolic orbits are examined.

In [1], the effect of relativistic precession of the axes of rotation of a spherical body (gyroscope) were examined from the point of view of a monadic method for defining the reference frame (RF) and the results obtained were used to find the exact expressions for the angular velocity of precession for the motion of a gyroscope along circular epiequatorial orbits in a Kerr field. However, it is of interest to determine the angle of precession also in the case of infinite motion of the gyroscope, since it is with infinite motion in particular that the meaning of the traditional definition of the comoving RF with respect to stationary stars is lost. In addition, the possible applied significance of this case, noted in the monograph [2] by Braginskii, should be kept in mind. The present work, which is a continuation of [1], is devoted to the solution, within the framework of the monadic approach, of the problem of finding the angle of precession for the angles of rotation of a spherical gyroscope for arbitrary ballistic motion of the gyroscope in a Schwarzschild field. For the definitions and notations of the monadic properties and other quantities, we follow everywhere [1, 3].

1. Choice of Comoving Frame of Reference

As shown in [1], the angular velocity vector of precession $\Omega^\mu$ relative to the comoving RF ($\bar{r}$) in which the center of mass of a gyroscope is at rest, is equal in magnitude and opposite in direction to the angular velocity vector for rotation of the comoving RF and has the following form:

$$\Omega^\mu = \gamma^2 (v^\mu/c) \left[ \left(1/2\right) e^{\alpha\beta} (v_\alpha v_\beta - v_\alpha v^\alpha) + \gamma^2 - v^\mu + \left(1/2\right) e^{\alpha\beta} (v_\alpha v_\beta + v_\alpha F^\beta - (v_\alpha/c) \delta^\beta v_\alpha) \right],$$

where $v_\mu$ is the three-dimensional velocity of motion of the basis of the comoving RF ($\bar{r}$) relative to some auxiliary RF ($\bar{r}$); $F^\mu_\nu$ and $O_\mu$ are the acceleration and angular velocity of rotation 3-vectors of the auxiliary RF ($\bar{r}$), $\gamma^2 = (1 - v^2/c^2)^{-1}$, $\epsilon_{\mu\nu\rho\lambda} = \sqrt{-1} g^\lambda_{\mu\nu} g^{\rho\lambda}$, and the indices for the three-dimensional quantities are raised with the help of the projector $b^{\mu\nu} = \gamma^{\mu\nu} - g^{\mu\nu}$ [1, 3].

Since (1) includes partial derivatives of the velocity $v_\mu$, the result depends not only on giving the world line of the representative point, but also on the behavior of world lines of neighboring reference bodies in the comoving RF, the specification of which to a large extent determines the quantitative significance of the effect [1]. We will choose a comoving RF ($\bar{r}$) in the vicinity of a world line represented by a point as follows. First,
let the 3-velocities of the reference bodies RF (\(\tau\)) be parallel in a section orthogonal to the direction of motion in the auxiliary RF (\(\tau\)). Second, let the instantaneous angular velocities of all points of this section be equal in magnitude. If we denote \(\gamma^{\mu\nu} = b^{\mu\nu} - v^{\mu}v^{\nu}/v^2\), then these requirements can be written in terms of the following equalities:

\[
\gamma^{\mu\nu}v^\nu = 0, \quad \gamma^{\mu\nu}v^\nu = -\varphi^\mu,
\]

where \(v\) is the magnitude of the 3-velocity \(v^\mu\), while \(\varphi^\mu\) is the 3-vector of the first curvature of the trajectory of the representative point in the auxiliary RF.

Since we assume ballistic motion for the representative point, using the equation of the geodesic in the monadic formalism [3] in order to express in Eq. (2) the spatial covariant derivative \(\nabla^\rho v^\mu\) in terms of \(v_\nu\) and \(\rho^\nu v_\nu\), in substituting \(\epsilon^{\mu\nu\rho}v^\nu = \epsilon^{\mu\nu\rho}v_\nu\rho\ln (1)\), we obtain that in the comoving RF (\(\tau\)) constructed according to (2) the angular velocity of precession has the form:

\[
\tilde{\Omega}^\rho = -\gamma^{\rho\nu}(\varphi/c)\gamma_\nu^\sigma\sigma^\nu + \gamma^{\rho\nu}(\varphi/c)\varphi_\nu^\sigma\sigma^\nu + (\varphi/c)\rho^\nu v_\nu.
\]

We note that conditions (2) determine in a narrow world tube a RF (\(\tau\)), which represents a model (known to be approximate) of the comoving RF, rigidly fixed to the carrier housing of the gyroscope. It is for this reason that in what follows we refer to \(\tilde{\Omega}^\mu\) (3) as the precession relative to the walls of the carrier, which is constantly oriented along its direction of motion in the auxiliary RF.

2. Angle of Precession in a Schwarzschild Field

Let the Schwarzschild metric be given in curvature coordinates \((x^0, r, \theta, \varphi)\). As the auxiliary RF (\(\tau\)), we will choose a stationary RF given by the vector field \(\tau^\sigma = (1 - 2r/m^2)^{-1/2}6^\sigma\), where \(m = Gm/c^2\). Assuming without loss of generality that the representative point moves in the plane \(\theta = \pi/2\) and introducing integrals of the energy \(a = \gamma(1 - 2r/m^2)^{1/2}\) and angular momentum \(\Sigma = \gamma r^2d\varphi/d\tau\), we transform (3) after simple calculations to the form

\[
\tilde{\Omega}^\rho = \frac{\rho^\sigma}{2r} \frac{\sigma + 1 - 2r/m^2}{\sigma - 1 + 2r/m^2}.
\]

From here it follows that the angular velocity of precession relative to RF (\(\tau\)) is oriented in an orthogonal orbital plane. Taking into account the fact that \(d\tau = \gamma^{-1}d\tau = r^2\Sigma^{-1}d\varphi\) and denoting \(r^{-1} = u(\varphi)\), we obtain from (4) the following expression for the increase in the angle of precession \(d\tilde{\alpha} = \tilde{\Omega}d\varphi/c^2:

\[
d\tilde{\alpha} = \frac{1}{2} \frac{\sigma + 1 - 2r/m^2}{\sigma - 1 - 2r/m^2} \frac{d\varphi}{c^2}.
\]

All quantities entering into (5) are expressed in terms of the eccentricity \(e\) and the inverse perihelion radius \(u\) of the orbit [4]. Thus, the energy integral \(a\) is obtained from the equation

\[
a^2 = 1 - \rho\mu - \frac{1 - \rho\mu}{1 + e} \frac{1 - e^2 - 4\rho\mu(1 - e)}{1 - e - 2\rho\mu},
\]

the function \(u(\varphi)\) has the form [5]

\[
u(\varphi) = \frac{u}{1 + e} \frac{1 - e^2 + 2e\text{sn}^2(P^2/2 + \varphi)}{1 + e - 2\rho\mu(3 - e)},
\]

where \(P^2 = 1 - 2\rho\mu(3 - e)/(1 + e)\), and the modulus \(k\) of the elliptic function equals

\[
k = 2(\rho\mu)^{1/2} [1 + e - 2\rho\mu(3 - e)]^{-1/2}.
\]

Thus, Eq. (5) permits calculating the angle of precession \(\Delta\tilde{\alpha}\) for motion of the gyroscope along an arbitrary section of the trajectory, if the orbital parameters \(u\) and \(e\) are known. For example, for parabolic orbits \(e = 1\). The total motion along such a trajectory corresponds (for \(\varphi_0 = 0\)) to a change in the coordinate \(\varphi\) from 0 to \(4K/P\) (where \(K\) is the total elliptic integral [6]). On the other hand, the total angle of precession \(\Delta\tilde{\alpha}\) par is obtained by integrating expression (5) along the entire trajectory. Retaining only terms of order \(\mu\), we obtain: