Local and nonlocal forms are obtained of the action induced by two-dimensional gravitation with torsion. It is shown that at the classical level this action can be rewritten in a form where there is no torsion. Cosmological applications and the possibility of the exact solvability are discussed.

It is well known that the multiplicative renormalizability of the interacting field theory in \( d = 4 \) curved space-time with torsion (for review see [1]) requires a nonminimal interaction of quantum fields with torsion. Generally speaking, upon the reduction of such a theory to \( d = 2 \), nonminimal interaction between the torsion and quantum fields remains also in two-dimensional space. General coordinate-free action of this type (without dimensional parameters) has the form:

\[
S = \int d^2x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} \Phi (\Box + \xi_1 T^2 + \xi_2 \bar{\psi} \gamma^a T_a) \Phi + \right. \\
+ \left. \bar{\psi} (\gamma^a \gamma^b \xi_3 T_a T_b) \psi \right\},
\]

where \( T^a = T^a_{\alpha\nu} \); \( T_{\alpha\beta\gamma} \) is the torsion tensor; \( \Phi, \psi \) are two-dimensional scalars and spinors; and \( \xi_1, \xi_2, \xi_3 \) are constants of nonminimal interaction between matter and torsion. We note that in \( d = 2 \), the torsion tensor has two independent components. Therefore it is completely determined by the vector \( T_\alpha \).

Using the classical action, the two-dimensional conformal anomaly is:

\[
T_A = -\frac{a}{4\pi} (R + a_1 T^2 + a_2 \bar{\psi} \gamma^a T_a),
\]

where \( R \) is the two-dimensional curvature without torsion; \( T^2 = T_{\alpha\beta} T^\alpha \); the constants \( a, a_1, a_2 \) (whose explicit form is not important to us) depend on the number of scalars and spinors; and \( a_1 \) and \( a_2 \) depend also on \( \xi_1, \xi_2, \xi_3 \).

The effective action is related to \( T_A \):

\[
\frac{2}{\sqrt{-\tilde{g}}} g_{\mu\nu} \frac{\delta \Gamma}{\delta g_{\mu\nu}} = T_A.
\]

We shall represent the metric in the form \( g_{\mu\nu} = e^{2\sigma} \tilde{g}_{\mu\nu} \), where \( \sigma \) is an arbitrary function of coordinates and \( \tilde{g}_{\mu\nu} \) has a fixed determinant. A simple calculation with this metric parametrization gives:

\[
\frac{\delta \Gamma_A}{\delta \sigma} = \frac{a}{4\pi} \sqrt{-\tilde{g}} \left[ (\tilde{\nabla}^2 + a_1 \tilde{T}^2 + a_2 \bar{\psi} \gamma^a \tilde{T}_a) + 2 \tilde{\square} \sigma \right],
\]

where \( \tilde{T}_\alpha \equiv T_\alpha \); \( \Gamma_A \) denotes the part of the effective action caused by the trace anomaly.

Integrating (3) one easily obtains:

\[
\Gamma_A = \frac{a}{4\pi} \int d^2x \sqrt{-\tilde{g}} \left[ (\tilde{\nabla}^2 + a_1 \tilde{T}^2 + a_2 \bar{\psi} \gamma^a \tilde{T}_a) \sigma + \sigma \tilde{\square} \sigma \right].
\]
This expression can be rewritten in a nonlocal form

\[ \Gamma_A = \frac{a}{4\pi} \int d^4x d^2y \left[ \frac{1}{4} (V - g R)_x G(x, y) (V - g R)_y + \right. \\
\left. + \frac{1}{2} \left[ V - g \left( a_1 T^2 + a_2 \nu^a T^a \right) \right]_x G(x, y) (V - g R)_y \right]. \]  

(5)

where \((\sqrt{-g})_x G(x, y) = \delta^2(x, y)\). For \(a_1 = a_2 = 0\), expressions (4) and (5) have been previously obtained in [2].

It is interesting that the action (5) can be rewritten in an equivalent local form

\[ \Gamma_A = \int d^2x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi + \right. \\
\left. + \frac{1}{2} \left[ a - \frac{1}{6} \right] \Phi \right] \left[ R + \frac{a}{a - \frac{1}{6}} \left( a_1 T^2 + a_2 \nu^a T^a \right) \right]. \]  

(6)

For \(a_1 = a_2 = 0\), expression (6) has been previously obtained in [3].

Actions (5) and (6) can be considered as models of induced quantum gravity with torsion. It is well known that induced two-dimensional gravity without torsion has interesting properties. Namely, in the conformal gauge, the Virasoro algebra is the unifying symmetry of the theory, while in the light-cone gauge, the algebra is the \(SL(2, \mathbb{R})\) current Kac–Moody algebra [4]. It would be interesting to investigate these properties for the case with torsion.

We consider the action (6) which can be written in a more convenient form

\[ \Gamma_A = \int d^2x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi + c_1 \Phi R + c_2 \Phi T^2 + c_3 \Phi \nu^a T^a \right]. \]  

(7)

where \(c_1, c_2,\) and \(c_3\) are constants.

The equation of motion \(\delta \Gamma_A / \delta T_\alpha = 0\) has the following obvious form

\[ T_\alpha = \frac{c_3 \nu^a \Phi}{2c_2}. \]  

(8)

Excluding torsion from the action (7) with the help of (8) we obtain

\[ \Gamma_A = \int d^2x \sqrt{-g} \left\{ \frac{1}{2} \left[ 1 + \frac{3c_2}{2c_2} \right] g^{\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi + c_1 \Phi R \right\}. \]  

(9)

We make the transformation \(g_{\mu\nu} = e^{2\sigma(\Phi)} \tilde{g}_{\mu\nu}\), where

\[ \sigma(\Phi) = \frac{3c_2}{16c_1 c_3} \Phi^2. \]  

(10)

Then (9) can be rewritten in the form

\[ \Gamma_A = \int d^2x \sqrt{-\tilde{g}} \left[ \frac{1}{2} \tilde{g}^{\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi + c_1 \Phi \tilde{R} \right]. \]  

(11)

In this way, the action with torsion (7) can be represented in the form of an action without torsion (11). Since for the action (11), it has been established that in the light-cone gauge it has a symmetry based on the \(SL(2, \mathbb{R})\) current Kac–Moody algebra and is exactly solvable, all these properties are valid also for the theory with torsion (7).

We shall now investigate simple cosmological applications. We consider the theory (4)-(6) as a model of two-dimensional gravitation with torsion. Then, possible two-dimensional Robertson–Walker models with torsion can be analyzed. The simplest choice is to take

\[ ds^2 = a^2(t)(dt^2 - dx^2), \quad T_\alpha(t) = (T(t), 0) \]  

(12)