Using the Riemann metric for event space, which leads to Newtonian mechanics at nonrelativistic velocities and not necessarily weak gravitational fields, the dynamics of relativistic particles in external gravitational fields are considered. Trajectories, laws of motion, and equations of light rays are found in homogeneous and Newtonian fields.

The motion of relativistic particles in gravitational fields determined experimentally or on the basis of field equations, occurs along extremes of the function \( \int L(x, dx) \) in event diversity. The function \( L(x, v) = L(x, dx)/dt \) depends on field potential and velocities. If \( L(x, dx) \) is of quadratic form \( g_{ik}dx^idx^k \) and the dependence of the metric on field potential \( \varphi(x) : g_{ik} = g_{ik}(x, \varphi(x)) \) is known, then the laws of motion and particle trajectories in various fields are determined by the equations of Riemann space geodesics. However, such a dependence is well known only in the "weak-field approximation" [1]. The metric tensor components may also be specified directly as functions of coordinates and time. In this case the relationship to the field potential may be implicit or not exist at all. These facts do not permit examination of relativistic particle dynamics in a homogeneous field, in a Newtonian gravitational field, and in other physically possible electrodynamic fields [2].

It was demonstrated in [3,4] that the only Riemann event space metric leading to Newtonian mechanics at nonrelativistic velocities and arbitrary (not obligatorily weak) gravitational field strengths is the metric

\[
ds^2 = \frac{\Lambda^4}{c^4} dx^\alpha dx^\beta - \frac{\Lambda^2}{c^2} (dx^2 + dy^2 + dz^2), \tag{1}\]

where \( \Lambda = \sqrt{c^2 + \varphi(x)} \).

In connection with this, the Lagrangian and Hamiltonian formulations of relativistic mechanics of a mass point in a gravitational field are determined by the functions

\[
L(x, dx) = -mc ds, \quad L(x, v) = -m \Lambda^2 \sqrt{1 - \frac{v^2}{\Lambda^2}}, \tag{2}
\]

\[
E = \frac{m \Lambda}{\sqrt{1 - \frac{v^2}{\Lambda^2}}}, \quad p = \frac{mv}{\sqrt{1 - \frac{v^2}{\Lambda^2}}}, \tag{3}
\]

\[
H(x, p) = \sqrt{p^2 \Lambda^2 + m^2 \Lambda^4}. \tag{4}
\]

The Hamilton–Jacobi equation has the form

\[
\left( \frac{\partial S}{\partial t} \right)^2 - \Lambda^2 (v S)^2 - m^2 \Lambda^4 = 0. \tag{5}
\]

The quadratic form of Eq. (1) also defines the form of the basic geometric optics equation.
\[
\left(\frac{\partial^2 \psi}{\partial t^2}\right) - \Lambda^2 (\nabla \psi)^2 = 0,
\]

where \( \psi \) is the eikonal.

We will consider motion in homogeneous and Newtonian gravitational fields.

**Homogeneous Gravitational Field**

We choose the z-axis along the direction of the field. Then \( \Phi = -az \). We solve Eq. (5) by the method of seperation of variables:

\[
S = -Et + x_0 x + y_0 y \pm \sqrt{\frac{E^2}{c^2 - az} - p_0^2 - p_0^2 c^2 (c^2 - az)} \frac{1}{az}.
\]

We choose the origin of the time coordinate at the moment when the particle is located at the point \( r = 0 \) and has a momentum \( p_x = p_{0x}, p_y = p_{0y}, p_z = p_{0z} < 0 \). Then from Eq. (7) we find the particle law of motion along the z axis

\[
t = \pm \frac{2}{\sqrt{a + \beta}} \left( \frac{E}{mc} \right) \{ F(\kappa, \kappa) \pm F(\varphi_0, \kappa) \}
\]

and the motion trajectory in the plane xz

\[
x = \pm 2 \sqrt{a + \beta} \frac{p_x c}{ma} \left\{ -E(\varphi, \kappa) \pm E(\varphi_0, \kappa) + \frac{\beta}{a + \beta} \left[ F(\varphi, \kappa) \pm F(\varphi_0, \kappa) \right] \right\}.
\]

Here the constants \( \alpha \) and \( \beta \) are related to the momentum \( p \) and the total energy \( E \) of the particle at time \( t = 0 \) as follows:

\[
\alpha = \sqrt{\frac{1}{4} \left( \frac{p_x}{mc} \right)^4 + \left( \frac{E}{mc^2} \right)^2 - \frac{1}{2} \left( \frac{p_x}{mc} \right)^2},
\]

\[
\beta = \alpha + \left( \frac{p_x}{mc} \right)^2, \quad p_x = p_{0x} + p_{0y}.
\]

\( F(\varphi, \kappa) \) and \( E(\varphi, \kappa) \) are elliptic integrals of the first and second kind [5], with arguments equal to

\[
\varphi = \arcsin \left[ 1 - \frac{1}{\alpha} \left( 1 - \frac{c^2}{c_0^2} \right) \right], \quad \varphi_0 = \arcsin \sqrt{1 - \frac{1}{\alpha}},
\]

\[
\kappa = \sqrt{\frac{\alpha}{\alpha + \beta}}.
\]

The minus sign in Eqs. (8), (9) corresponds to motion of the mass point in the direction opposite the direction of the field. Then \( 0 \geq z \geq z_{\text{min}} \), where \( z_{\text{min}} = (1 - \alpha)(c^2/a) \) is the particle reversal point. The plus sign corresponds to \( p_z > 0 \) and \( z \geq z_{\text{min}} \).

As follows from [3], the results presented are valid for motion of a noninteracting particle in a reference system moving with constant acceleration \( a \).

A homogeneous gravitational field may be considered in a limited region of space. Let \( |z| \ll c^2/a \). For simplicity we set \( p_{0z} = 0 \), and from Eq. (8) we have

\[
z = \frac{1}{2} g \left[ 1 + \frac{1}{2} \left( \frac{p_x}{mc} \right)^2 \right] \left( \frac{mc^2}{E} \right)^2.
\]

Equation (10) gives the law for "fall" of a relativistic particle in a homogeneous gravitational field \( g \). It may be used, in particular, for estimation of "sag" of relativistic particle trajectories in accelerators. In the nonrelativistic limit Eqs. (8)-(10) reduce to the known results of Newtonian mechanics.

We will now consider the law of light ray propagation in a field \( \phi = -az \). Choosing the plane xz parallel to the wave vector \( z \), from Eq. (6) we find the eikonal

\[
\psi = -\omega t + x \chi_x \pm \int \sqrt{\frac{\omega^2}{c^2 - az} - \chi_x^2} \, dz.
\]

Differentiating Eq. (11) with respect to \( \chi_x \), we obtain the equation of a light ray:

\[
x = \pm \frac{1}{2a} \frac{\omega^2}{\chi_x^2} \arcsin \left[ 2 \left( \frac{c \chi_x}{\omega} \right)^2 - 1 \right] \pm \frac{c^2}{a} \sqrt{\left( \frac{\omega}{c \chi_x} \right)^2 - 1}.
\]