GROUP FORMULATION OF PERTURBATION THEORY
FOR PARTICLES IN AN ELECTROMAGNETIC FIELD

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If the Hamilton—Jacobi equation for a particle in an electromagnetic field [1]

\[ g^{ik} \left( \frac{\partial S}{\partial x^i} + \frac{e}{c} A_i(x) \right) \left( \frac{\partial S}{\partial x^k} + \frac{e}{c} A_k(x) \right) = m^2 c^2 \]

(\(i = 0, 1, 2, 3\)) admits of a group of transformations of the space \(E^4(x)\), then the trajectories of a particle subjected to operations from the admissible group go over again into trajectories with the very same electromagnetic-field potential. Instead of the function of the action dependent on the variables \(x^i\), let us consider the function \(S(x, y)\) by replacing the potential \(A_i(x)\) in (1) by \(\frac{e A_i(x)}{\partial S/\partial y}\), we obtain that the equation

\[ g^{ik} \left( \frac{\partial S}{\partial x^i} + \frac{e}{c} A_i(x) \frac{\partial S}{\partial y} \right) \left( \frac{\partial S}{\partial x^k} + \frac{e}{c} A_k(x) \frac{\partial S}{\partial y} \right) = m^2 c^2 \]

(2)

contains solutions with separable variables

\[ S(x, y) = S(x) + \lambda y \]

and \(S(x)\) satisfies (1) with the potential \((eA/c)A_i(x)\), i.e., trajectories of a particle with the potential \((eA/c)A_i(x)\) can be obtained from (2) for any \(\lambda\), and if (2) admits of a transformation group of the space \(E^5(x, y)\), then operations from the admissible group will transfer the trajectories of particles with one value of \(\lambda\) into trajectories with another value of \(\lambda\). Taking into account that (2) admits of the group of transformations of the space \(E^5(x, y)\) only for a definite class of potentials, it is necessary to introduce several additional variables \(y^\alpha \) (\(\alpha = 0, 1, \ldots, m - 1\)), and to impose the following requirements on the equation for \(S(x, y)\): The equation should be homogeneous in all the derivatives, should contain a solution with separated variables \(S(x, y) = S_1(x) + S_2(y)\) so that \(S_1(x)\) is a solution of (1), the equation admits of a simply transitive transformation group \(G\) for the space \(E^4 + m(x, y)\) such that its reciprocal group \(\hat{G}\) has an exact representation in \(E^5(x)\) [2]. This latter condition isolates the variables \(x^i\) and means that there exists a subgroup \(H\) for which the \(x^i\) are invariant while \(H\) is a simply transitive transformation group in the space \(E^m(y)\). The equation for \(S(x, y)\) is invariant relative to \(H\), hence it can be written in terms of the invariants \(H\), \(i\), \(x^i\), \(\partial/\partial x^i\), \(\partial/\partial y^\alpha\) \((\partial^\alpha = \omega_3^i(y)(\partial/\partial y_3)\) are generators of the group algebra \(H)\); taking [1] into account, we obtain


It follows from the condition \( S(x, y) = S_1(x) + S_2(y) \) that the system of equations \( \dot{\omega}_\alpha S = \lambda_\alpha \), \( \lambda_\alpha \) a constant, has a nonzero solution but this means that
\[
\lambda_\alpha e^{\mu} = 0, \tag{4}
\]
where \( C_{\beta \mu}^\alpha \) are structural constants of the group algebra \( \mathbb{H} \):
\[
[C, \omega] = C, \quad [\omega, C] = \omega, \quad [\omega, \omega] = 0. \tag{5}
\]
Formally, (3) is obtained from (1) by replacing \( A_1(x) \) by the differential expression \( A_1^\alpha(x) \dot{\omega}_\alpha S \) and \( m^2 c^2 \) by \( m^2 c^2 + g^{\alpha \beta}(x) (\dot{\omega}_\alpha S) (\dot{\omega}_\beta S) \); let us require that commutation relations be satisfied after such a replacement, i.e., the expression \( m^2 c^2 + g^{\alpha \beta}(x) (\dot{\omega}_\alpha S) (\dot{\omega}_\beta S) \) should behave as a constant, and the commutator of the operations \( (\partial/\partial x^i) + (e/c)A_1^\alpha(x) \dot{\omega}_\alpha \) should yield an expression corresponding to the electromagnetic field tensor.

This means that \( g^{\alpha \beta} \) is independent of \( x^i \), while \( g^{\alpha \beta} \dot{\omega}_\alpha \dot{\omega}_\beta \) commutes with all \( \dot{\omega}_\nu \)
\[
\{e, g^{\alpha \beta}\} = \{g^{\alpha \beta}, e\} c = 0, \tag{5'}
\]
and the potentials \( A_1^\alpha(x) \) are related by the condition
\[
A_1^\alpha(x) c_i = A^\alpha_i(x) = 0. \tag{6}
\]

It follows from (4) and (5) that \( \lambda_\alpha g^{\alpha \beta} \dot{\omega}_\beta \) commutes with all the \( \dot{\omega}_\alpha \); by using the arbitrariness in the selection of the coordinate system in \( E^m(y) \) and of the basis in the group algebra \( \mathbb{H} \), it can be considered that \( \lambda_\alpha g^{\alpha \beta} \dot{\omega}_\beta \) is proportional to \( \omega \), \( \omega = \partial/\partial y^\rho \), then we obtain for \( S_1(x) \)
\[
g^{\alpha \beta} \left( \frac{\partial S_1}{\partial x^i} + \frac{\lambda e}{c} A_1^\alpha(x) \right) \left( \frac{\partial S_1}{\partial x^j} + \frac{\lambda e}{c} A_1^\beta(x) \right) = m^2 c^2 + \frac{\lambda e}{c} \dot{\omega} \cdot \dot{\omega}, \tag{7}
\]
\[
i.e., \text{ the electromagnetic field is determined by the potential } (e\lambda/c)A_1^\alpha(x).\]

**Equations of Motion**

The action function given by (3) governs the geodesic Riemann space with the length element
\[
ds^2 = g_{ij} dx^i dx^j - g_{ij} \xi^i \xi^j, \tag{6}
\]
by taking into account that the Lagrange function for the free particle is proportional to \( ds/dt \) [1], we select \( L \) in the form
\[
L = -mc^2 \rho, \quad \rho = \left[ 1 - \frac{1}{c^2} \left[ (e \psi + \frac{e}{c} A^\alpha \psi - \frac{e}{c} A^\alpha \psi) \right] \right]^{1/2},
\]
where \( \psi = dx/dt \), \( \psi^\alpha = dy^\alpha/dt \), \( (\psi^\alpha, A^\alpha) \) is a vector with the components \( g^{1\alpha} A^\alpha \). The generalized momenta on energies are given by the expression
\[
P^\alpha = g_{\alpha \beta} \psi^\beta, \quad P = \frac{m \psi}{\rho} + \frac{e}{c} A^\alpha \psi, \tag{7}
\]
\[
u = \frac{m \psi}{\rho} + \frac{e}{c} A^\alpha \psi, \quad \nu^\alpha = \frac{e}{c} A^\alpha \psi - \frac{e}{c} A^\alpha \psi, \quad \psi = \frac{e}{c} A^\alpha \psi - \frac{e}{c} A^\alpha \psi
\]
\[
H - e \varphi \omega = c \sqrt{m^2 c^2 + g^{\alpha \beta} \omega \omega + \left( P - \frac{e}{c} A^\alpha \right)^2}.
\]
Taking into account that \( \psi \psi/\rho = \psi \{c^2 (E - e \varphi \omega\omega) \}^{-1} \), this expression can be integrated as a momentum \( p = \psi \psi/\rho \) and the equation of motion can be written as
\[
\frac{dp}{dt} = e \omega \left[ E^* + \frac{1}{c} \left[ H^* \right] \right], \quad H^* = \text{rot } A^*, \quad E^* = - \frac{1}{c} \frac{\partial A^*}{\partial t} - \varphi^*,
\]
\[
\frac{d \omega_s}{dt} = \frac{\partial H}{\partial \omega_s} c_p \omega_s. \tag{8}
\]