The Grossman—Peres classical electron model is not explicitly a relativistic invariant. The relativistic invariance of the Grossman—Peres model is proved in this paper by the direct construction of Poincaré-group generators from integrals of the motion of the model under consideration. The generators found afford the possibility of obtaining also an expression for the 4-vector of coordinate-time.

The problem of an adequate description of particles with spin 1/2 occurred with the discovery of spin on an electron. Dirac soon solved this problem successfully in quantum mechanics. Up to now there is no standard analog of the Dirac equation in classical mechanics. One of the most satisfactory models of a classical relativistic particle with spin is the model proposed by Grossman—Peres [1]. The equations of motion in this model are found by using a formal replacement of operators and their commutators in the Heisenberg equations of motion of the Dirac electron by classical dynamic quantities and their Poisson brackets, respectively. As the initial quantum equations of motion, the classical equations obtained in this manner have no explicitly covariant form.

The purpose of this paper is to prove the relativistic invariance of the Grossman—Peres equations. To do this it is first necessary to construct the Poincaré-group generators from the integrals of motion of the Grossman—Peres equations, except that in this case the equations of motion in the new variables retain their initial form [2]. Moreover, additional considerations assuring the correct nature of the transformation of the known quantities are also necessary for uniqueness in the selection of the generators. The Poincaré-group generators have been constructed in [1] also; however, not all the generators proposed there were integrals of the motion, and the generators of the spatial rotations did not assure the vector nature of the transformations introduced in the theory of vectors.

1. Canonical Hamiltonian Formalism

As has been shown in [3], the Grossman—Peres model admits the construction of a canonical formalism if "internal" complex coordinates \( \psi = (\psi_1, \psi_2, \psi_3, \psi_4^T \) (\( \tau \) is the transpose symbol) and their conjugate "momenta" \( \psi^+ = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^* ) \) are introduced in place of the "external" coordinates \( r \) and their conjugate momenta \( p \). The equations of motion are hence obtained from a variational principle, i.e., from the requirement of an extremum of the integral.
and will have the form

\[ \dot{r} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial r}, \quad \dot{\gamma} = \frac{1}{i\hbar} \frac{\partial H}{\partial \gamma^*}, \quad \dot{\gamma}^* = -\frac{1}{i\hbar} \frac{\partial H}{\partial \gamma}, \]

where H is the Hamiltonian function and \( \hbar \) is the unit of measurement of the action.

The Poisson brackets \{f, g\} of the dynamic quantities f and g are defined by the relationship

\[ \{f, g\} = \frac{\delta f}{\delta r} \frac{\delta g}{\delta p} - \frac{\delta f}{\delta p} \frac{\delta g}{\delta r} + \frac{1}{i\hbar} \left( \frac{\delta f}{\delta \gamma^*} \frac{\delta g}{\delta \gamma} - \frac{\delta f}{\delta \gamma} \frac{\delta g}{\delta \gamma^*} \right). \]

By means of this definition the equations of motion become standard for classical mechanics

\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}. \]

If the expression

\[ H = \gamma^* a^\dagger p c + \phi^* \beta \gamma mc^2, \]

is taken as H, where \( \hat{a} \equiv (\hat{a}_x, \hat{a}_y, \hat{a}_z) \), \( \beta \) are the Dirac 4 x 4 Hermitian matrices satisfying the anticommutation relationships

\[ \hat{a}_i \hat{a}_j + \hat{a}_j \hat{a}_i = 2i \delta_{ij}, \quad \hat{a}_i \beta + \beta \hat{a}_i = 0, \quad i, j = x, y, z. \]

I is the unit matrix, \( \delta_{ij} \) is the Kronecker delta, m is the electron mass, and

\[ a = \gamma^* a^\dagger, \quad \beta = \gamma^* \beta^\dagger, \quad S = \frac{1}{4i} \phi^* \hat{a} \times \hat{a}^\dagger, \quad T = \frac{1}{2} \phi^* \beta \hat{a}^\dagger, \]

is introduced, then we obtain the Grossman-Peres equations for a free electron by using (2)-(7)

\[ \frac{d\gamma}{dt} = \gamma, \quad \frac{dp}{dt} = 0, \quad \frac{d\phi}{dt} = 4p \times S + 4mT, \quad \frac{dS}{dt} = -4p \cdot T. \]

Here and below the system of units with \( c = \hbar = 1 \) is taken.

The equations for an electron in a field is obtained if \( p \) is replaced in (5) by \( p - eA \) and the term \( e\phi \) is added to (5) from the right, where \( A \) is a vector potential, \( \phi \) is a scalar potential, and \( e \) is the charge on the electron.

Using the properties of the Dirac matrices, the Poisson braces of the quantities introduced in (7) are easily computed:

\[ \{\gamma_k, \gamma_l\} = 4\gamma_{kl} S_l, \quad \{S_k, S_l\} = \gamma_{kl} S_l, \]

\[ \{\gamma_k, \beta\} = 4T_k, \quad \{T_k, T_l\} = \epsilon_{klj} S_j, \]

\[ \{\phi, S_l\} = 0, \quad \{S_k, S_l\} = \epsilon_{klj} \gamma_j, \]

\[ \{\gamma_k, \gamma_l\} = \delta_{ij}, \quad \{T_k, S_l\} = \epsilon_{klj} T_j, \]

\[ \{T_k, \gamma_l\} = \delta_{ij}, \quad (i, j, k = x, y, z), \]

which agree with those postulated in [1].

2. Canonical Transformations

By analogy with the canonical transformations for real variables [4], canonical transformations can be constructed for complex variables [5]. The fact is hence used that the integrand in the integral (1) for which the variational principle is applied is determined to the accuracy of the total derivative of some function F with respect to the time \( dF/dt \). The function F is the generating function of the canonical transformation, i.e., the new \( Q_j, P_j, \Psi, \Psi^+ \) and the old \( q_j, p_j, \psi, \psi^+ \) variables can be connected by one of the following relationships:

\[ p_j = \frac{\partial F}{\partial q_j}, \quad P_j = -\frac{\partial F}{\partial Q_j}, \quad \phi^* = \frac{1}{i\hbar} \frac{\partial F}{\partial \phi^*}, \quad \Psi^+ = -i\hbar \frac{\partial F}{\partial \Psi}, \]

(10.1)