Electrostatic and Optical Resonances of Cylinder Pairs

R. C. McPhedran and W. T. Perrins
Department of Theoretical Physics, School of Physics, University of Sydney, N.S.W. 2006, Australia

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Abstract. We study the polarizability $\alpha(\sigma)$ of a pair of identical cylinders, each of dielectric constant $\sigma$ and immersed in a medium of unit dielectric constant. In the case of non-touching cylinders, the spectrum of resonant solutions is shown to be discrete, whereas when the cylinders touch it is continuous, and corresponds to a branch-cut of $\alpha(\sigma)$. This behaviour is compared with that of the spectrum of $\varepsilon(\sigma)$, the dielectric constant of an infinite array of cylinders. We derive a simple expression for $\alpha(\sigma)$, and use it to obtain $\varepsilon(\sigma)$ for a dilute array of touching cylinder pairs. The resulting formula is shown to have properties differing from those of the widely-used Maxwell-Garnett and effective medium theories, and consequently it may prove useful in studies of solar-selective columnar cermets.

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In [1] the spectrum of resonant solutions was studied for the problem of calculating the effective transport coefficient of the square and hexagonal arrays of circular cylinders. In agreement with results of Bergman [2, 3], the spectrum was shown to be discrete in the case of lattices of non-touching cylinders. However, in order to explain the asymptotic results of O'Brien [4] for arrays of touching cylinders in terms of the spectrum of resonances, it was convenient to postulate a continuous spectrum in this particular case. The continuous spectrum was inferred to represent a branch-cut of the function $\varepsilon(\sigma)$, $\varepsilon$ being the effective transport coefficient of the array and $\sigma$ being that of the cylinder material (both being normalized with respect to the transport coefficient of the matrix material).

Here, we will study the spectrum of resonant solutions in connection with a much simpler problem than that of the transport coefficient of an infinite array. Using the specific language of the dielectric constant to exemplify the general quantity "transport coefficient", we will discuss the polarizability $\alpha(\sigma)$ of a pair of identical cylinders, each having dielectric constant $\sigma$ and immersed in a medium of unit dielectric constant. We will study the spectrum of resonant solutions for $\alpha(\sigma)$, in the cases of non-touching and touching cylinders. In the former case, we will show that the spectrum is discrete, and will obtain a simple expression for the value of $\sigma$ corresponding to each resonant solution. In the latter case, we will show that the spectrum is continuous, and corresponds to a branch-cut of $\alpha(\sigma)$. The position of the branch-cut will be shown to depend on the direction of the applied electrostatic field $E_0$ with respect to the line of centres of the cylinder pair.

We also consider the dielectric constant of a rectangular array of cylinders, with the cylinders touching along one axis of periodicity (say the x-axis) but not along the other. We present numerical evidence to show that [just as for $\alpha(\sigma)$] the position of the branch-cuts of $\varepsilon(\sigma)$ depends on the orientation of $E_0$ with respect to the x-axis. We also compare numerical results for $\varepsilon(\sigma)$ for touching rectangular arrays of cylinders with approximate formulae due to Ninham and Sammut [5]. We show that, while their formulae are accurate for $\sigma$ close to unity, they cannot be used when this quantity is large.

Finally, we use our expression for the polarizability of a pair of touching cylinders in the Clausius-Mossotti formula [6], and obtain an expression for the dielectric
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Fig. 1. (a) The two cylinders of radius \( a \), and centre-to-centre separation \( 2l \), with the coordinate system \( xOy \) indicated. (b) The two coaxial cylinders resulting from the transformation \( t = 1/z \). The rectangular coordinate system \( uOv \) and the polar coordinate system \( (r, \theta) \) are indicated.

constant of a dilute array of touching cylinder pairs. We compare this new expression with the widely-used Maxwell-Garnett and effective medium formulae in a particular case, and show that it yields complex refractive indices with an imaginary part which is significant at considerably longer wavelength than for the other two formulae. This feature may well make the new formula attractive in studies of the optical properties of cermets [7]. It is readily interpreted in terms of the resonance spectra of the three theories.

1. Resonant Solutions for a Pair of Cylinders

Consider a pair of cylinders, each of dielectric constant \( \sigma \) and radius \( a \), placed in a medium of unit dielectric constant with their axes separated by a distance \( (2l) \) (Fig. 1a). Take a Cartesian rectangular coordinate system \( Oxy \), with the \( Ox \) axis intersecting both cylinder axes, and the origin \( O \) at a distance \( d \) from the axis of the right cylinder, where

\[ d = l + \sqrt{l^2 - a^2}. \]

We seek to find a distribution of potential \( V(x, y) \) satisfying Laplace’s equation, the electrostatic boundary conditions (continuity of \( V \) and of the normal component of the electric displacement vector) at the surface of each cylinder and which can exist in the absence of an applied electrostatic field. Such a distribution will be called a resonant solution of the electrostatic problem for the cylinder pair. Define

\[ z = x + iy, \]

\[ t = u + iv \]

then

\[ t = \frac{1}{z} = (x - iy)/(x^2 + y^2). \]

Binns and Lawrenson [8] show that the transformation (4) maps the cylinder pair of Fig. 1a onto the coaxial cylinders of Fig. 1b, the common axis passing through the point \( C(c, 0) \) in the \( (u, v) \) plane where

\[ c = d/(d^2 - a^2). \]

Measuring polar coordinates \( (r, \theta) \) with respect to \( C \) as origin, the surfaces of the transformed cylinders have as their equations \( r = r_1 \) and \( r = r_2 \), where

\[ r_1 = a/\sqrt{2l^2 - a^2(l - \sqrt{l^2 - a^2})} \]

and

\[ r_2 = a/\sqrt{2l^2 - a^2(l + \sqrt{l^2 - a^2})}. \]

If \( n \) is any positive integer, we can write down two resonant solutions of order \( n \), symmetric about \( \theta = 0 \). The first of these is

\[ V(r, \theta) = \left( \frac{r}{r_2} \right)^n \cos(n \theta) \quad \text{for} \quad r \leq r_2, \]

\[ = \left[ \frac{r_1}{r_2} \right]^n \left( \left( \frac{r_1}{r} \right)^n + \frac{r}{r_1} \right) \cos(n \theta)/(r_1 + 1) \quad \text{for} \quad r_2 \leq r \leq r_1, \]

\[ = \left( \frac{r_1}{r} \right)^n \cos(n \theta) \quad \text{for} \quad r \geq r_1. \]

Here,

\[ \tau_n = \left( \frac{r_2}{r_1} \right)^n = \frac{1 + \sigma_n}{1 - \sigma_n}, \]

the dielectric constant of each cylinder being

\[ \sigma = \sigma_n = \frac{\tau_n - 1}{\tau_n + 1} < 0. \]