X⁴-IN Variant 5-Dimensional Gravitation Theory

B. G. Aliyev, Yu. S. Vladimirov, and V. S. Nedopekin

1. In [1, 2] a 5-dimensional gravitation theory was analyzed with the fifth coordinate introduced by a method similar to that of chronometric invariants of A. L. Zel'manov. In the above approach two conditions were imposed to obtain a spherically symmetric solution: 1) the gauge-invariant derivatives of geometric quantities with respect to the fifth coordinate must vanish (this corresponds to the usual cylinder condition); 2) the gauge-invariant antisymmetric tensor $F_{\alpha \beta}$ must also vanish (there is in fact no electromagnetic field). It appears that more natural constraints can be adopted to obtain the same results if the 5-dimensional gravitation theory is constructed by introducing the fifth coordinate with the aid of a method similar to the kinemetric invariants method [3, 4].

2. Similarly as in [3], the quantities invariant with regard to transformations of the fifth coordinate are called $x^4$-invariant ($x^4$-inv.),

$$x'^\mu = x'^\mu(x^\nu)$$  \hspace{1cm} (1)

if they are covariant with regard to transformations of the other four coordinates,

$$x^{\nu'} = x^{\nu'}(x^0, x^1, x^2, x^3, x^4),$$ \hspace{1cm} (2)

where $\mu = 0, 1, 2, 3$. The following components of any tensor possess this property:

$$A_{\nu'}^{\mu} = \frac{\alpha}{G^{\mu}}$$ \hspace{1cm} (3)

where $G_{\mu\nu}$ is a metric tensor of a 5-dimensional manifold. (The subscripts $\alpha$ and $\beta$ assume the following values: 0, 1, 2, 3, 4).

3. The following $x^4$-inv. combinations can be formed from the components of the metric tensor:

$$G_{\mu}^{\nu} = \tau^0 = 0, \quad G^{\mu} = G^{\mu\nu} - G^{\mu\nu} G^{\nu\nu} G^{\nu\nu}, \quad G_{\mu}^{\nu} = G_{\nu}^{\mu}$$ \hspace{1cm} (4)

The components of the metric tensor are now represented in the following form:

$$G_{\mu\nu} = \tau^0 \tau^0 + \tilde{G}_{\mu\nu} G^{\mu} G^{\nu},$$ \hspace{1cm} (5)

where $\tau^A = G^{4A}/G^{44}$; $\tau^A = 1/\sqrt{G^{4}}$ and thus the following relations are valid:

$$\tau^A \tau^A = 1; \quad \tau^A \tilde{G}_{\mu\nu} = \tau^A \tilde{G}_{\nu\mu} = 0.$$ \hspace{1cm} (6)

One then has

$$\tau^0 = 0; \quad \tau^4 = \frac{1}{G^{44}} \equiv \frac{1}{K}, \quad \tau^i = K, \quad \tau^4 = \frac{G^{44}}{K};$$

$$\tilde{G}_{44} = \tau^4 \tau^4; \quad \tilde{G}_{i} = \tau^i; \quad \tilde{G}_{4} = - \frac{G^{44}}{K^2};$$

$$\tilde{G}_{44} = \tilde{G}_{4i} = \tilde{G}_{i4} = \tilde{G}_{ii} = 0.$$ \hspace{1cm} (7)


©1976 Plenum Publishing Corporation, 227 West 17th Street, New York, N Y 10011 No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of the publisher A copy of this article is available from the publisher for $15.00
4. Ordinary derivatives of $x^4$-inv. quantities are not $x^4$-inv. expressions; they maintain, however, the $x^4$-inv. of the following differential operators:

$$\frac{\partial A_{\mu \nu \lambda}^4}{\partial x^4} \rightarrow \partial^4 A_{\mu \nu \lambda}^4 \equiv \nabla^4 A_{\mu \nu \lambda}^4 +$$

$$+ \sum_{m \text{ terms}} \nabla^m A_{\mu \nu \lambda}^4 \times \sum_{n \text{ terms}} \nabla^n A_{\mu \nu \lambda}^4,$$

where $\nabla^\mu = \tau_{\mu \nu} - \nabla \mu B_{\mu \nu}$. $B_{\mu \nu} \equiv (1/K)(\partial K/\partial x^\mu)$ is a $x^4$-inv. quantity;

$$\frac{\partial}{\partial x^\nu} A_{\mu \nu \lambda}^4 \times \partial^4 A_{\mu \nu \lambda}^4$$

$$+ \sum_{m \text{ terms}} \nabla^m A_{\mu \nu \lambda}^4 \sum_{n \text{ terms}} \nabla^n A_{\mu \nu \lambda}^4,$$

where $E_{\mu \nu}^4 = (1/2)^{\rho \sigma \lambda \mu \nu}$ are four-dimensional interrelation coefficients.

It follows from (8) and (9) that similarly as in the theory of kinematic invariants all $x^4$-inv. differential operators depend on whether the differentiated quantity is covariant. For $A = 4$, the operators $S x^4 \nu$ vanish; for $A = \mu$ these operators correspond to covariant differentiation of the 4-dimensional interrelation $E_{\mu \nu}^4$.

5. Instead of $A_{\mu \nu \lambda \alpha \beta \gamma \delta}$ one substitutes the components $E_{\mu \nu}$ into (8) and one obtains the following $x^4$-inv. geometric quantities:

$$\tilde{A}_{\alpha \beta}^4 = -\frac{1}{2} \partial^4 \tilde{g}_{\alpha \beta}^4 = -\frac{1}{2} \left( \tau^A \tilde{g}_{\alpha \beta}^4 - \tilde{N}_{\alpha}^\beta \tilde{g}_{\beta}^4 - \tilde{N}_{\beta}^\alpha \tilde{g}_{\alpha}^4 \right),$$

$$\tilde{A}_{\alpha \beta}^4 = \frac{1}{2} \partial^\alpha \tilde{g}_{\beta}^4 = \left( \tau^A \tilde{g}_{\alpha \beta}^4 + \tilde{N}_{\alpha}^\beta \tilde{g}_{\beta}^4 + \tilde{N}_{\beta}^\alpha \tilde{g}_{\alpha}^4 \right).$$

(10)

It follows from (9) that

$$\partial^\nu \tilde{g}_{\alpha \beta}^4 = \partial^\nu \tilde{g}_{\alpha \beta}^4 = 0$$

The commutator of the operators of the $x^4$-inv. four-dimensional differentiation operating on an $x^4$-inv. vector $A_\sigma$

$$[\partial^\alpha, \partial^\beta] A_\sigma = \tilde{M}_{\mu \nu \alpha \beta} A_\gamma = \left( \frac{\partial \tilde{E}_{\alpha \mu}^4}{\partial x^\gamma} - \frac{\partial \tilde{E}_{\beta \mu}^4}{\partial x^\gamma} + \tilde{E}_{\alpha \nu}^4 \tilde{E}_{\beta \nu}^4 - \tilde{E}_{\alpha \nu}^4 \tilde{E}_{\beta \nu}^4 \right) A_\gamma$$

(11)

results in an $x^4$-inv. four-dimensional curvature tensor. Moreover, one also has

$$[\partial^\alpha, \partial^\beta] A_\alpha = (B_{\alpha} \tilde{A}_{\beta} + B_{\beta} \tilde{A}_{\alpha} - B_{\alpha} \tilde{A}_{\beta}).$$

(12)

6. Let us consider now the geodesics in the 5-dimensional space,

$$\frac{d^2 x^A}{dl^2} = - P_{\beta \gamma}^A \frac{dx^\beta}{dl} \frac{dx^\gamma}{dl},$$

(13)

where the four-dimensional coherency coefficients assume their usual form,

$$P_{\beta \gamma}^A = \frac{1}{2} G^{AD} \left( G_{BD,\xi} + G_{DL \nu} - G_{BL \xi} \right),$$

(14)

and the five-dimensional square $dl^2$ of the interval can be written as

$$dl^2 = G_{AB} dx^A dx^B = \left( \frac{G^{AD} dx^A}{V G^{AD}} \right)^2 + \left( G_{\nu \gamma} - G_{\nu \gamma} G^{4^4} G_{4^4} \right) dx_{4} dx_{4},$$

(15)

We now introduce the $x^4$-inv. components of $dx^\mu$ and of 5-dimensional velocities,