COVARIANT FORMULATION OF THE WEAK CONSERVATION LAW IN THE
GENERAL THEORY OF RELATIVITY WITHOUT THE ENERGY–MOMENTUM
OF THE GRAVITATIONAL FIELD

V. N. Tunyak

A systematic covariant formulation is given of the weak conservation law in the
general theory of relativity for arbitrary coordinate transformations without
introducing the unsatisfactory definition of the energy–momentum of the gravita-
tional field.

As is well known, one of the fundamental problems in the general theory of relativity
(GTR), even as long ago as the beginning of its development [1–7], has been the problem of
determining the localization of the energy–momentum of the gravitational field (EMGF). Two
fundamentally different approaches to the solution of this problem have been proposed. Ac-
cording to the point of view of Lorentz [2] and Levi-Civita [3], later adopted in [8–11] as
well, there exists an EMGF tensor proportional to the Einstein tensor

\[ T^\nu_\mu = - \frac{1}{x} G^\nu_\mu. \]  

(1)

Obviously, in view of the Einstein equations

\[ G^\mu_\nu = R^\mu_\nu - \frac{1}{2} g^\mu_\nu R = xT^\mu_\nu. \]  

(2)

the corresponding total tensor of the energy–momentum of the gravitational field and matter
reduces to zero:

\[ T^\nu_\mu = T^\nu_\mu + T^\nu_\mu = 0. \]  

(3)

The inadequacy of the hypothesis (1) of Lorentz and Levi-Civita [2, 3] was immediately
pointed out by Einstein, who indicated that from Eq. (3) "...it is impossible to derive those
conclusions that we are used to making from conservation laws" [7]. Einstein then asserted
that a "...more suitable definition of the components of the energy of the gravitational

V. I. Lenin Belorussian State University. Translated from Izvestiya Vysshikh Uchebnykh
field" than the introduction of a noncovariant EMGF tensor is impossible [5]. The fundamental unsatisfactory feature of this approach is the contradiction between the principle of general covariance and the noncovariant definition of the EMGF. Thus, attempts to resolve the problem of the localization of the energy-momentum of the gravitational field lead to two apparently equally unsatisfactory results — to the generally covariant tensor (1), the introduction of which deprives the conservation laws of real physical meaning and is essentially a simple reformulation of the Einstein equations, and to the noncovariant quasitensor, which implies an arbitrary limitation of the covariance of GTR. It should be remarked that, although in the process of the further development of GTR many new formulations of the EMGF were proposed (tetrad, γ-matrix, quaternion, bimetric) [12-18], the above-mentioned alternative remains valid. The point is that, independently of one or another specific choice of gravitational potentials in GTR, we arrive either at the tensor (1) or at the noncovariant EMGF quasitensor, so that only the form of this noncovariance is different in the various formulations [12-20].

In the situation that has been created the solution of the problem of EMGF in GTR, and correspondingly the elimination of the above-mentioned alternative, can be achieved only by eliminating the concept of EMGF, which is illegitimate in GTR, and subsequently considering GTR as the geometrical study of the properties of the noneuclidean structure of space-time that are due to matter. The EMGR problem in GTR does not exist, since EMGR itself does not exist! From the point of view of this approach, developed in the work of Eddington [21] and M. F. Shirokov [22, 23], the EMGF "...which we need if we want to retain the formal appearance of a conservation law, do not form a tensor; they should be considered as a mathematical function, and not as a representation of some world relations possessing physical meaning" [21]; the EMGF are "...only an intuitive, artificial means of interpreting the curvature of Riemannian space-time in the language of forces and energies acting in a fictitious flat space-time instead of the objectively real Riemannian space-time of GTR" [22]. The subsequent accomplishment of this program in classical (nonquantum) GTR requires its realization on the basis of the variational principle of GTR. This paper is devoted to the solution of the latter problem.

We will start from the variational principle, based on the tetrad formulation of GTR (TFGTR), which has been used to include GTR in the general theory of gauge fields [14-16, 24-28]:

\[ \delta \int (\Lambda_m + \Lambda_g) d^4x = 0, \] (4)

where \( \Lambda_m = \sqrt{-g} L_m, \Lambda_g = \sqrt{-g} L_g \), and \( L = -R/2\kappa \). We take the Lagrangian of nongravitational matter in the form

\[ L_m = L_m(Q_A, \nabla_a Q_A), \] (5)

where the potentials \( Q_A \) are local components of tensors and spinors, and

\[ \nabla_a Q_A = \eta^A_a \{ \partial_i Q_A + f_{(a)(b)} B^B Q_B A^{(a)(b)} \}, \] (6)

the so-called invariant derivative; \( f_{(a)(b)} \) are constants which are generators of the Lorentz transformation, and \( A^a_{(\mu)(\nu)} \) are the coefficients of the Ricci rotation in the Riemannian space \( V_a \). For this choice of \( L_g, \) the tetrad Einstein tensor has the form

\[ x^{-1} G_{(a)^{(s)}} = \nabla_u U^{u(a)} - t_{(a)^{u}}, \] (7)

where

\[ U^{u}_{(a)} = U[\nabla^u_{(a)}] = -D^u_{(a)} - K^u_{(a)} Q_{(a)} \] (8)

is the so-called antisymmetric superpotential;

\[ t_{(a)^{u}} = D^u_{(a)} Q_{(a)} + K^{[u}_{(a)} \nabla_{(a)} Q_{(a)}^{(s)} \] (9)

\[ D^u_{(a)} = \partial \partial_{(a)} h^u_{(a)} - \partial K_{(a)}^{u^{(s)}} - \nabla_{(a)} K^{u^{(s)}} (a) + \{a\} K^{u^{(s)}} (a), \] (10)

\[ K_{(a)^{(s)}} = \partial L_{g}^{(s)} \partial \partial_{(a)} h^u_{(a)} ; \] (11)