QUASICLASSICAL COHERENT TRAJECTORY STATES
OF A SPINLESS RELATIVISTIC PARTICLE
IN AN ARBITRARY ELECTROMAGNETIC FIELD

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Approximate (quasiclassical) solutions of the Klein–Gordon equation for a charge in an
arbitrary electromagnetic field are constructed. One of the decisive properties of these
solutions is that the average quantum–mechanical values of the coordinates and momenta
in these states are the exact solutions of the classical relativistic Hamiltonian equations.

In this work we construct the quasiclassical states of a relativistic particle, described by the Klein–
Gordon equation in an arbitrary electromagnetic field such that the average quantum mechanical values of the
coordinates $x (x_1, x_2, x_3)$ and momenta $p (-i\hbar \partial_{x_1}, -i\hbar \partial_{x_2}, -i\hbar \partial_{x_3})$ in these states are the exact solutions of the
classical Lorentz equations. By analogy to [1], these are called coherent trajectory states (CTS). An explicit
construction of the CTS and an estimate of their accuracy are presented below and the completeness of the sys-
tem of CTS in the class of solutions of the Klein–Gordon equation for positive-frequency wave functions is
proven. The method for constructing these solutions is based on a version of the complex WKB method — the
complex shoot theory of Maslov [2, 3].

1. Basis Notation

The Klein–Gordon equation for a particle in an arbitrary electromagnetic field has the form

$$[(i\hbar \partial_t - eA_0)^2 - (-i\hbar \gamma - eA)^2 - m^2c^2] \Psi = 0,$$

(1)

where $A_0 = A_0 (x, t)$, $A = A(x, t)$ are the electromagnetic field potentials in the Lorentz gauge $\partial_t A_0 + \text{div} A = 0$.

We denote by $\Psi_+ (x, t)$ the positive-frequency normalized solution of the Schrödinger-like equation

$$\hat{\mathcal{L}}_+ \Psi_+ = 0, \quad \hat{\mathcal{L}}_+ = i\hbar \partial_t - \hat{H} (x, p, t), \quad \int \Psi_+ \Psi_+^* d^3x = 1,$$

(2)

where the quantum operator $\hat{H} (x, p, t)$, the nonlocal $\hbar^{-1}$ pseudodifferential operator, is defined by the equation

$$\hat{H} (x, p, t) \Psi_+ (x, t) = (2\pi \hbar)^{-3/2} \int \exp \left[ -\frac{i}{\hbar} (p, x) \right] \hat{H} (x, p, t) \Psi_+ (p, \hbar) d^3p.$$

(3)

Here $\tilde{\Psi}_+ (p, \hbar)$ is the Fourier transform of the function $\Psi_+ (x, t),

$$\tilde{\Psi}_+ (p, \hbar) = (2\pi \hbar)^{-3/2} \int \exp \left[ -\frac{i}{\hbar} (p, x) \right] \Psi_+ (x, t) d^3x.$$

The function $\hat{H} (x, p, t)$ is the classical relativistic Hamiltonian [4],

$$\hat{H} (x, p, t) = eA_0 (x, t) + \sqrt{[e\mathcal{L} - eA (x, t)]^2 + m^2c^4} = eA_0 (x, t) + E_+ (x, p, t).$$

(4)

We shall assume known the general solution $x (t), p (t)$ of the classical Lorentz equations (taken in the
Hamiltonian form with the Hamiltonian (4)),

$$\dot{x} = \mathfrak{H}_p (x, p, t), \quad \dot{p} = -\mathfrak{H}_x (x, p, t), \quad x (0) = x_0, \quad p (0) = p_0.$$

(5)

We denote by $\mathfrak{H} (x, p)$ the $9 \times 9$ matrix in variations for system (5), which we write in the form of $(3 \times 3)$ blocks

$$\mathfrak{H} (x, p) = \begin{pmatrix}
-H_{xp} (x, p, t) & -H_{xp} (x, p, t) & H_{pp} (x, p, t) \\
H_{xp} (x, p, t) & H_{xp} (x, p, t) & H_{pp} (x, p, t)
\end{pmatrix}.$$
where, e.g.,
\[ H_{xx} = \| \partial_x \partial_x f(x, p, t) \|, \quad H_{xp}(x, p, t) = \| \partial_x \partial_p f(x, p, t) \|. \]

The blocks of the matrix \( \mathcal{H}(x(t), p(t)) \equiv \mathcal{H}(t) \), calculated at the trajectory points \( x(t), p(t) \) of system (5), have the form

\[
H_{pp} = [c^2 \delta_{ij} - \dot{x}_i \dot{x}_j] \mathcal{E}_{+}^{-1}(t),
\]

\[
H_{xp} = H_{xp}^* = e \dot{x}_i (x(t), \nu_x A_j(t)) - e c^2 \partial_x \partial_i A(t) \| c E_+(t) \|^{-1},
\]

\[
H_{xx} = e \| c^2 E_+(t) \| \partial_x \partial_x A_0(t) + \delta c \sum_{k=1}^3 \delta x_k \partial_x A_k(t)\]

\[ - c E_+(t) \dot{x}(t), \partial_x \partial_x A(t) - e (\dot{x}(t), \partial_x A(t)) \| (x(t), \partial_x A(t)) \| [c^2 E_+(t)]^{-1}. \]

where \( E_+(t) = E_+(x(t), p(t), t) \) and \( I \) is the unit \((3 \times 3)\) matrix.

All of the results of the nonrelativistic case [1] are valid for the Schrodinger equation (2) after the corresponding replacement of the nonrelativistic Hamiltonian by Eq. (4). That is, the specified classical trajectories of an electron \( x(t), p(t) \) and the complex-valued nonsingular \((3 \times 3)\) matrices \( B(t) \) and \( C(t) \) are solutions of the system in variations (with respect to the trajectory \( x(t), p(t) \)),

\[
\left( \begin{array}{c} B \\ C \end{array} \right) = e \mathcal{H}(t) \left( \begin{array}{c} B \\ C \end{array} \right), \quad B(0) = \text{diag} (b_1, b_2, b_3),
\]

\[
C(0) = I,
\]

where \( \mathcal{H}(t) \) is determined by Eqs. (6) and (7) and \( b_k \) are complex numbers, \( \text{Im} b_k > 0, k = 1, 2, 3 \), uniquely determining the complete set of CTS for the relativistic equation (1) according to the scheme discussed in [1].

2. System of Relativistic Coherent Trajectory States

We shall present the final results here without the detailed calculations analogous to those of [1]. The complete set of CTS \( \Psi_{+9}(x, t, \hbar) \) is generated by the approximate (as \( \hbar \to 0 \)) symmetries of \( \hat{a}_k(t) \) of the operator \( L_+ \) from the equation

\[
\Psi_{+9}(x, t, \hbar) = \prod_{k=1}^3 (\sigma_k - 1) \left[ \hat{a}_k^+ (t) - a_k^*(t) \right] \Psi_{+0}(x, t, \hbar),
\]

where \( \Psi_{+0}(x, t, \hbar) \) is the normalized vacuum CTS solution of the WKB type,

\[
\Psi_{+0}(x, t, \hbar) = N(h) \Phi(t) \exp \left[ i \hbar^{-1} S_+(x - x(t), t) \right],
\]

\[ N(h) = [(\pi \hbar)^{-3} \text{Im} b_1 \text{Im} b_2 \text{Im} b_3]^{1/4} \]

with the complex phase

\[ S_+(x, t) = \int_0^t \mathcal{A}(\dot{x}(t), \dot{x}(t), t) dt + (x - x(t), p(t)) + (x - x(t), Q(t) x - x(t))/2. \]

Here \( \mathcal{A} \) is the relativistic Lagrangian [4] on the trajectory

\[ \mathcal{A}(\dot{x}, \dot{x}, t) = - mc^2 \sqrt{1 - \beta^2} - e [A_0(t) - c^{-1} (\dot{x}, A(t))], \]

\[ c_0 = \dot{x}, \quad A_1(t) = A_1(x(t), t), \]

the amplitude \( \Phi(t) = [\det C(t)]^{-1/2}, Q(t) \) is a symmetric \((3 \times 3)\) matrix, \( Q(t) = B(t)C^{-1}(t) \) with positive imaginary part.

The annihilation \( \hat{a}_k(t) \) and creation \( \hat{a}_k^+(t) \) operators satisfy exactly the standard commutation relations and are determined by the complex column vectors \( w_k(t), Z_k(t) \) of the matrices \( B(t) \) and \( C(t) \) from the equations

\[
\hat{a}_k(t) = (2 \hbar \text{Im} b_k)^{-1/2} [(Z_k(t), \hat{p}) - (w_k(t), x)],
\]

\[
\hat{a}_k^+(t) = (2 \hbar \text{Im} b_k)^{-1/2} [(Z_k^*(t), \hat{p}) - (w_k^*(t), x)],
\]

while the eigenvalues \( \alpha_k(t) \) of the operators \( \hat{a}_k(t) \) are, respectively,

\[ z_k(t) = (2 \hbar \text{Im} b_k)^{-1/2} [(Z_k(t), p(t) - (w_k(t), x(t))]. \]

\[ \hat{a}_k(t) \Psi_{+0}(x, t, \hbar) = z_k(t) \Psi_{+0}(x, t, \hbar). \]