NONLINEAR OSCILLATIONS OF COMBUSTION VELOCITY OF POWDER

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The combustion of powder, where the surface temperature $T_s$ depends on the pressure $p$ and the initial temperature $T_0$, is studied under the condition of sinusoidally varying pressure. The nonlinearity of the heat conduction equation together with the dependence of the combustion velocity $u$ and the surface temperature on both the pressure and initial temperature affects the zeroth harmonic and gives rise to higher harmonics in the combustion velocity and the temperature of the powder. The present paper considers the case of nonlinear resonance, when the frequency of the pressure fluctuations is close to the natural vibration frequency of the powder. It has been shown that an autonomous oscillatory regime of combustion is possible under constant pressure.

1. FORMULATION OF PROBLEM AND BASIC EQUATIONS

It was shown in [1] that nonstationary processes of powder combustion with variable surface temperature can be studied with the same method as that used by Zel'dovich [2,3] for the case of constant temperature $T_s$. If all inertias except the heat transfer in the condensed phase are neglected, it can be shown that the surface temperature $T_s$ of the powder, together with the combustion velocity $u$ during nonstationary combustion are determined by the instantaneous values of the pressure and internal temperature gradient at the powder surface $f$. The relations $u(f, p)$ and $T_s(f, p)$ can be obtained from the laws of steady-state combustion $u(T_0, p)$ and $T_s(T_0, p)$ by replacing the initial temperature $T_0$ using the following expression in terms of $u$, $T_s$, and $f$

$$ u^o(T_s^o - T_0) $$

(1.1)

which is valid for the stationary regime ($\chi$ is the thermal conductivity of the powder).

The linearized problem of powder combustion under sinusoidally varying pressure was studied in [4]. In the present paper we study nonlinear effects including nonlinear resonance and autonomous oscillations.

Let $p^o$ be the mean pressure and $u^o$, $T_s^o$ be the corresponding combustion velocity and surface temperature in the steady-state combustion at pressure $p^o$.

Now we define the dimensionless variables

$$ \vartheta = \frac{T - T_0}{T_s^o - T_0}, \quad \xi = \frac{u^o}{\chi x}, \quad \tau = \frac{(u^o)^2}{\chi} t, $$

$$ v = \frac{u}{u^o}, \quad \eta = \frac{p}{p^o}, \quad \varphi = \frac{\vartheta}{\vartheta_i} |\xi| $$

(1.2)

where $x$ is the coordinate ($x < 0$), and $t$ is the time. In the inertial region, i.e., the heated layer of the condensed phase, we have the following heat conduction equation,

$$ \frac{\partial \vartheta}{\partial \xi} = \frac{\partial \vartheta}{\partial \xi^2} - v \frac{\partial \vartheta}{\partial \xi} $$

(1.3)

together with the boundary conditions

$$ \vartheta |_{\xi=0} = \vartheta, \quad \vartheta |_{\xi=\infty} = 0 \quad (\theta = \frac{T_s - T_0}{T_s^o - T_0}) . $$

The system of equations (1.3), (1.4) becomes complete when supplemented by the following relations for the combustion velocity and surface temperature in terms of the pressure and gradient

$$ v = v(\varphi, \eta), \quad \vartheta = \vartheta(\varphi, \eta) $$

(1.5)

together with the expression for the pressure which will be chosen to have the cosinusoidal form

$$ \eta = 1 + \frac{1}{2} (\cos \gamma t + \sin \gamma t) $$

(1.6)

where $(2)^{1/2}H$ is the pressure amplitude.

The system (1.3)-(1.6) theoretically determines the time dependence of combustion velocity and powder temperature, but it is not possible to solve this system for arbitrary functions (1.5). However, it is possible to expand the solution in a series of powers of the small pressure amplitude.

2. COMBUSTION VELOCITY AND TEMPERATURE DISTRIBUTION: THIRD ORDER APPROXIMATION

A periodic force applied to a nonlinear vibratory system gives rise to oscillations at all harmonics. If the amplitude of the force is small, the correction to the constant component and the second harmonic is proportional to the square of the amplitude, and the first harmonic, as compared with the linear approximation, has a third order correction in terms of the amplitude (see [5,6,7]).

The steady-state of combustion at pressure $\eta = 1$ is described by the solution

$$ \vartheta = \vartheta^o, \quad v = 1, \quad \vartheta = 1, \quad \varphi = 1 . $$

(2.1)

When the pressure varies according to (1.6), the higher order approximation will be sought in the form

$$ v(\tau) = 1 + \varphi_1 \cos \gamma \tau + \varphi_2 \sin \gamma \tau + \varphi_3 \cos 2\gamma \tau + \varphi_4 \sin 2\gamma \tau, $$

$$ \vartheta(\tau) = 1 + \vartheta_1 \cos \gamma \tau + \vartheta_2 \sin \gamma \tau + \vartheta_3 \cos 2\gamma \tau + \vartheta_4 \sin 2\gamma \tau, $$

$$ \varphi(\tau) = 1 + \varphi_1 \cos \gamma \tau + \varphi_2 \sin \gamma \tau + \varphi_3 \cos 2\gamma \tau + \varphi_4 \sin 2\gamma \tau, $$

$$ \vartheta(\xi_1) = e^{1+\varphi_1} \left[ 1 + \vartheta_1(\xi) \cos \gamma \tau + \vartheta_2(\xi) \sin \gamma \tau + \vartheta_3(\xi) \cos 2\gamma \tau + \vartheta_4(\xi) \sin 2\gamma \tau \right] $$

(2.2)

(2.3)

(2.4)
We substitute expression (2.2) for the velocity and (2.5) for the temperature into the heat conduction equation, multiply the trigonometric functions, retaining only the zeroth, 1st, and 2nd harmonics, and then we equate coefficients of like trigonometric functions. This yields five ordinary differential equations for the unknowns \( \theta_{10}, \theta_{15}, \psi_2, \theta_{20}, \) and \( \theta_{25} \).

The complex notation
\[
\theta_n = \theta_{0n} + i\theta_{1n}, \quad \nu_n = \nu_{0n} + i\nu_{1n}, \quad \psi_n = \psi_{0n} + i\psi_{1n}, \quad \vartheta_n = \vartheta_{0n} + i\vartheta_{1n} \quad (n = 1, 2) \quad (2.6)
\]

enables us to reduce the five differential equations to the following two complex and one real differential equation
\[
\begin{align*}
\theta_{1,\prime}'' + \theta_{1,\prime} - i\psi_1 &= \nu_1 - \nu_0 \theta_{1,\prime} + \\
+ \nu_1 (\psi_0'' + \psi_0') + \nu_0^2 \psi_1 + \psi_0' + \\
+ \nu_0^2 \nu_1 (\theta_{0,\prime} - \theta_0''), \\
\psi_{1,\prime}'' + \psi_{1,\prime} &= \nu_0 \psi_1 + \psi_0 (\theta_{0,\prime} - \theta_0''), \\
\theta_{2,\prime}'' + \theta_{2,\prime} - 2i\psi_2 &= \nu_2 - \nu_0 \theta_{2,\prime} + \\
&+ \nu_0^2 \nu_1 (\theta_{0,\prime} - \theta_0''),
\end{align*}
\quad (2.7)
\]

where the prime denotes differentiation with respect to \( \xi \), and the bar indicates a complex conjugate. In the linear approximation, (2.7) has the solution [4]
\[
\begin{align*}
\theta_1 &= \theta_{10} + A_1 (e^{i2\xi} - 1), \quad A_1 = \theta_{10} - i\nu_1 / \gamma \\
\psi_1 &= \psi_{10} + A_1 (e^{i2\xi} - 1), \quad A_1 = \psi_{10} + i\nu_1 / \gamma
\end{align*}
\quad (2.10)
\]

where
\[
\begin{align*}
z_1 &= \frac{1}{2} (\gamma / R_1 - 1) + iR_1, \\
R_1 &= \left(\frac{1}{2} (\sqrt{16\gamma^2 + 1} - 1)\right)^{1/2}.
\end{align*}
\quad (2.11)
\]

It is worth noting that
\[
z_1^2 + z_1 - i\gamma = 0, \quad 4R_1^4 + R_1^2 - \gamma^2 = 0. \quad (2.12)
\]

Having the expression for \( \theta_2 (\xi) \) in the first approximation, we can find \( \psi_2 (\xi) \) and \( \vartheta_2 (\xi) \) in the second order approximation from equations (2.8), (2.9). The constant component is found to be
\[
\psi_2 = \psi_2 + \frac{1}{2\gamma} \left[ A_1 \psi_1 (1 + z_1) \times \\
(\frac{1}{2} \psi_1' (1 + z_1)) \right] (e^{i2\xi} - 1) - A_1 \psi_1 (1 + z_1) (e^{i2\xi} - 1). \quad (2.13)
\]

The spatial distribution of the second harmonic of the temperature in the condensed phase is
\[
\theta_2 = \theta_{20} + \left( A_2 - \frac{\nu_{01}^2}{4\gamma} \right) \times \\
\times (e^{i2\xi} - 1) + \frac{\nu_{01}^2}{2\gamma} (1 + z_1) (e^{i2\xi} - e^{i2\xi}) \quad (2.14)
\]

where
\[
\begin{align*}
A_2 &= \theta_{20} - i\nu_1 / 2\gamma, \\
z_2 &= \frac{1}{2} (\frac{\gamma}{2R_2} - 1) + iR_2, \\
R_2 &= \left[ \frac{1}{2} \left( \frac{\gamma}{2R_2} + 1 - 1 \right) \right]^{1/2}
\end{align*}
\]

Finally, substituting (2.10), (2.13) into the right-hand side of (2.7) and solving this equation, we obtain the amplitude of the first harmonic with an accuracy up to and including third order terms as given below
\[
\begin{align*}
\theta_1 &= \theta_{10} + \left( A_1 + \frac{\nu_{01}^2}{4\gamma} \right) (e^{i2\xi} - 1) - \\
&- \frac{A_1 \nu_{01}^2}{1 + 2\gamma} \frac{\psi_1^2 - \psi_1 (1 + z_1)}{2\gamma} \times \\
&\times \left( e^{i2\xi} - e^{i2\xi} \right) + \frac{\nu_{01}^2}{2\gamma} (1 + z_1) (e^{i2\xi} - e^{i2\xi}) \times \\
&\times \left[ A_2 - \frac{\nu_{01}^2}{2\gamma} (1 + z_1) - \frac{i\gamma \nu_{01}^2}{4\gamma^2} \psi_1', \right] \\
A_3 &= \frac{i\gamma \nu_{01}^2}{4\gamma^2} \psi_1, \\
&- \frac{1}{\gamma} [A_1 \vartheta_1 (1 + z_1) - A_2 \vartheta_1 (1 + z_1)]. \quad (2.16)
\end{align*}
\]

It follows from (2.4), (2.5) that
\[
\begin{align*}
\varphi_1 &= (1 + \nu_1) \theta_0 + \frac{\partial \theta_0}{\partial z} |_{z=0}, \\
f_2 &= w_2 + \psi_2 + \frac{\partial \psi_2}{\partial z} |_{z=0}, \\
q_2 &= \vartheta_2 + \frac{\partial \vartheta_2}{\partial z} |_{z=0}.
\end{align*}
\]

The derivatives in these expressions can be calculated from (2.16), (2.13), and (2.14). We then obtain the following three algebraic equations connecting the nine quantities: the constant components, the amplitudes of the first two harmonics of the gradient, the surface temperature, and the combustion velocity
\[
\begin{align*}
\varphi_1 &= \theta_1 (1 + \nu_1) + A_2 \frac{\nu_{01}^2}{2\gamma} (1 + z_1) - \\
&- \frac{A_1 \nu_{01}^2}{1 + 2\gamma} \frac{\psi_1^2 - \psi_1 (1 + z_1)}{2\gamma} \times \\
&\times \left[ A_2 - \frac{\nu_{01}^2}{2\gamma} (1 + z_1) - \frac{i\gamma \nu_{01}^2}{4\gamma^2} \psi_1', \right] \\
&\times \left[ A_2 - \frac{\nu_{01}^2}{2\gamma} (1 + z_1) - \frac{i\gamma \nu_{01}^2}{4\gamma^2} \psi_1', \right] \\
&\times \left[ A_2 - \frac{\nu_{01}^2}{2\gamma} (1 + z_1) - \frac{i\gamma \nu_{01}^2}{4\gamma^2} \psi_1', \right] \\
&f_2 = w_2 + \psi_2 + \frac{1}{2} \left( \psi_1 \vartheta_1 + \theta_1 \vartheta_2 \right), \\
&\varphi_3 = \vartheta_3 + \frac{i\nu_{01}^2}{2} + z_2 \left[ A_3 - \frac{\nu_{01}^2}{4\gamma} \frac{\psi_1^2}{2\gamma} (1 + z_1) \right]. \quad (2.19)
\end{align*}
\]

A further six equations can be obtained by expanding the functions \( \psi (\varphi, \eta) \) and \( \theta (\varphi, \eta) \) in Taylor series up to the third order terms. After multiplication of the trigonometric functions and equating coefficients of like harmonics, we obtain,
\[
\begin{align*}
v_1 &= \frac{\partial \varphi}{\partial \varphi} q_1 + \frac{\partial \varphi}{\partial \eta} q_1 + \frac{\partial \psi}{\partial \varphi} (\psi_1 f_2 + \bar{\psi} w_2) + \\
&\times \frac{\partial \psi}{\partial \varphi} \frac{\partial \eta}{\partial \varphi} (\eta_1 f_2 + \bar{\psi} w_2) + \frac{1}{8} \left( \frac{\partial \psi}{\partial \varphi} q_1 \psi_1 + \frac{\partial \psi}{\partial \varphi} \psi_1 \bar{q}_1 \right) \times \\
&\times (2q_1 \bar{q}_1 q_1 + \psi_1 \bar{\eta}_1 q_1) + \frac{\partial \psi}{\partial \varphi} \frac{\partial \eta}{\partial \varphi} (2q_1 \bar{q}_1 q_1 + \psi_1 \bar{\eta}_1 q_1), \quad (2.20)
\end{align*}
\]

where
\[
\begin{align*}
w_2 &= \frac{\partial \varphi}{\partial \varphi} f_2 + \frac{1}{4} \left( \frac{\partial \psi}{\partial \varphi} \bar{q}_1 \psi_1 + \right.
\end{align*}
\]