ON RADIATIVE HEAT TRANSFER IN A PLANE LAYER OF AN ABSORBING MEDIUM

N. A. Rubtsov
Zhurnal Prikladnoi Mekhaniki i Tekhniceoskoi Fiziki, No. 5, pp. 58-67, 1965

Recently there has arisen increased interest in the study of radiative heat transfer between geometrically simple systems, both as autonomous problems and as elements of more complex problems.

Problems of this kind have been treated by many authors [1-11], who have considered gray, diffusely emitting and absorbing boundaries and gray non-scattering media. In most cases these investigations were restricted either to the derivation of approximate formulas for the net radiative flux, without an exact analysis of the temperature distribution in the layer [5-7], or to numerical computation [1-4]. In the latter case, with the exception of [8], which contains a numerical analysis for the case of optical symmetry, no attempt was made to analyze the effect of the optical properties of the boundaries on the temperature field in the layer.

These papers can be divided into two groups according to the method of analysis used. The first group includes papers based on the integral equations of radiative transfer, with the corresponding integral analytical methods [1, 2]. Similar in nature are [3, 4] which use the slab method, applicable to electrical-analog computation, as well as a recent paper [6] based on probability methods.

The second group of papers [5-7] is based on the so-called differential methods. Of particular interest is [7], which develops these methods to an advanced degree. In several papers the problem of radiative transfer is analyzed in conjunction with more complex problems (e.g., [10, 11]).

In the present work we shall attempt to carry out an approximate analytical study of problems connected with radiative heat transfer in a plane layer of an absorbing, emitting, non-scattering gray medium with temperature-independent optical properties. The layer is bounded by two parallel, diffusely emitting and diffusely reflecting, isothermal, gray planes.

The paper presents the fundamental formulation of the problem, which consists in: (a) the determination of the net heat flux on the basis of given temperature distribution (direct formulation), and (b) the determination of the temperature distribution on the basis of given distribution of the net radiative heat source per unit volume and boundary temperatures (inverse formulation). The analysis is based on integral methods appropriate to the integral equations which represent the net total and hemispherical radiation flux densities [12].

The integral equations of radiative transfer in a radiating system of arbitrary configuration are [12]

\[
E(M) = A(M) \int A(N) E_0(N) \Gamma(M, N) dF_N + \int A(M) \times \eta_0(P) Z(M, P) dV_P - A(M) E_0(M) \tag{1}
\]

\[
= \eta(M) \eta_0(M) - \eta(M) \int A(N) E_0(N) \Gamma_1(M, N) dF_N \tag{M \in F},
\]

\[
- \eta(M) = \text{div } F_{4\text{}} + \eta_0(M) = 4E_0(M) = 4\eta T^4(M),
\]

Here \(\eta(M)\) is the net total radiative heat source per unit volume at the point \(M\), \(E_{4\text{}}\) is the radiation flux vector, \(E(M)\) is the net hemispherical radiative heat flux at the point \(M\), \(\eta(M)\) is the total black-body source function, \(E_0(M)\) is the hemispherical black-body radiation flux density, \(\times(M)\) is the volumetric absorption (emission) coefficient of the medium at the point \(M\), and \(A(N)\) is the emissivity of the surface at the point \(N\). In Eqs. (1) and (2) the resolving kernels \(Z(M, P), \Gamma(M, N)\) have the physical meaning of shape factors between the fixed point \(M\) and the generic volume element \(P\) and surface element \(N\):

\[
Z(M, P) = L(M, P) + \int R(P) \times \eta_0(P) Q_0(M, N) Z(N, P) dF_P,
\]

\[
\Gamma(M, N) = Q(M, N) + \int R(P) \times \eta_0(P) Q_0(M, P) \Gamma(P, N) dF_P,
\]

\[
L(M, P) = \exp \left(-\frac{\sigma T^4}{4\pi M P}\right),
\]

\[
Q(M, N) = \exp \left(-\frac{\sigma T^4}{4\pi M N}\right).
\]

In an analogous manner we can represent the resolving kernels \(Z_1(M, P), \Gamma_1(M, N)\) by the equations

\[
Z_1(M, P) = L_1(M, P) + \int R(N) Q_1(M, N) Z_1(N, P) dF_N,
\]

\[
\Gamma_1(M, N) = Q_1(M, N) + \int R(P) Q_1(M, P) \Gamma(P, N) dF_P,
\]

\[
L_1(M, P) = \exp \left(-\frac{\sigma T^4}{4\pi M P}\right),
\]

\[
Q_1(M, N) = \exp \left(-\frac{\sigma T^4}{4\pi M N}\right).
\]

Here \(r\) is the distance between \(M\) and \(P\) or \(M\) and \(N\), and \(\theta_M, \theta_N\) are the angles between the generic ray and the normals to the surface elements at \(M\) and \(N\). To apply the integral equations (1), (2) to the specific radiating system under consideration, we formally introduce into these equations the geometrical characteristics of the system. In particular, if we take into account the fact that the surfaces \(F_1, F_2\), which constitute the boundary \(F\), are nonconcave, we find that the self-irradiation shape factors of the surfaces vanish, \(Q_1(P_1, N_1) = Q_1(P_2, N_2) = 0\) (here \(P_1, P_2\) are the intermediate reflecting area elements on the surfaces \(F_1, F_2\), respectively), and that reflection takes place between the two outer bounding surfaces only.
One can easily show that the integral terms which appear in (1) and (2) can be expressed in explicit form in terms of the geometrical-optical parameters $Q(M, N, N_0), Q_1(M, P, P_0), L_q(M, P)$. Thus in Eq. (2) for $\eta(M)$ we have

\[
\begin{align*}
\{ & \int_{{\phi}} A(N_1)E_0(N_1) \Gamma_1(M, N_1) dF_{N_1} + \frac{1}{D_{12}} \int_{{\phi}} A(N_1)E_0(N_1) \left[ Q_1(M, N_1) + \int_{{\phi}} R(N_2) Q_1(M, N_2) Q(N_2, N_1) dF_{N_2} \right] dF_{N_1}, \\
& \int_{{\phi}} A(N_1)E_0(N_1) \left[ Q_1(M, N_1) + \int_{{\phi}} R(N_2) Q_1(M, N_2) Q(N_2, N_1) dF_{N_2} \right] dF_{N_1},
\end{align*}
\]

(3)

\[
\begin{align*}
\{ & \int_{{\phi}} A(N_1)E_0(N_1) \Gamma_1(M, N_1) dF_{N_1} + \frac{1}{D_{12}} \int_{{\phi}} A(N_1)E_0(N_1) \left[ Q_1(M, N_1) + \int_{{\phi}} R(N_2) Q_1(M, N_2) Q(N_2, N_1) dF_{N_2} \right] dF_{N_1}, \\
& \int_{{\phi}} A(N_1)E_0(N_1) \left[ Q_1(M, N_1) + \int_{{\phi}} R(N_2) Q_1(M, N_2) Q(N_2, N_1) dF_{N_2} \right] dF_{N_1},
\end{align*}
\]

(4)

The effect of multiple reflections is represented here by the geometrical-optical relation

\[
D_{12} = 1 - \int_{{\phi}} R(N_1) R(N_2) Q(N_1, N_2) Q(N_2, N_1) dF_{N_2} dF_{N_1}.
\]

Substituting the above geometrical-optical parameters, analogous to radiation shape factors, into (2), we obtain for $\eta(M)$ an integral equation in a form which can be used directly in further calculations.

In the course of the calculations we use the exponential integral $K_N(x)$, which is characteristic of transfer processes in absorbing media, the optical depth $\tau$, and the optical thickness $\tau_0$ of the layer,

\[
K_N(x) = \frac{1}{\tau} \int_0^x e^{-x} \mu^{-1} \frac{d\mu}{\mu}, \quad \tau = \int_0^\infty \frac{\mu}{\tau} d\mu, \quad \tau_0 = \int_0^\infty \frac{\mu}{\tau} d\mu.
\]

In the case when $F_1$ and $F_2$ are optically homogeneous and isothermal, the expression for the net radiative heat source per unit volume is

\[
-\eta(\tau) = \frac{dE}{d\tau} = 4E_0(\tau) - 2\int_0^\tau E_0(\zeta) K_1(\tau - \zeta) d\zeta -
\]

\[
-2 \int_0^\tau E_0(\zeta) \Psi(\tau, \zeta) d\zeta =
\]

\[
\frac{2}{1 - 4R_1R_2K_2(\tau_0)} \left\{ A_1E_0(\zeta)(K_2(\tau) + 2R_1 K_2(\tau_0) K_2(\tau_0 - \tau)) +
\right.
\]

\[
\left. + A_2E_0(\zeta)(K_2(\tau_0 - \tau) + 2R_1 K_2(\tau_0) K_2(\tau)) +
\right.
\]

\[
2R_1 K_2(\tau_0 - \tau) \int_0^\tau E_0(\zeta) K_2(\tau_0 - \tau) d\zeta +
\]

\[
+ 2R_1 K_2(\tau) \int_0^\tau E_0(\zeta) K_2(\tau_0 - \tau) d\zeta\right]\right] +
\]

\[
\left[ K_2(\tau_0 - \tau) \int_0^\tau E_0(\zeta) K_2(\tau) d\zeta + K_2(\tau) \int_0^\tau E_0(\zeta) K_2(\tau_0 - \tau) d\zeta\right]\right]_0^\tau.
\]

This expression agrees with an analogous expression for $\eta(\tau)$ in [13], obtained by a direct radiative heat flux balance.

Integrating the right and left sides of (6) term by term, and using the rules for integration under an integral sign, we obtain an expression for the net hemispherical radiative heat flux. Following [14], we represent the latter, as well as Eq. (6), in the form

\[
E(\tau) = 2A_1(\tau) E_{0,1} - 2A_2(\tau) E_{0,2} + 2\int_0^\tau E_0(\zeta) \Psi(\tau, \zeta) d\zeta -
\]

\[
-2 \int_0^\tau E_0(\zeta) \Psi(\tau, \zeta) d\zeta,
\]

\[
\frac{dE}{d\tau} = 4E_0(\tau) - 2A_1'(\tau) E_{0,1} -
\]

\[
-2A_2'(\tau) E_{0,2} - 2\int_0^\tau E_0(\zeta) Z(\tau, \zeta) d\zeta,
\]

(7)