ON THE EQUATIONS OF NONEQUILIBRIUM GAS FLOWS

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Introduction

In investigating the flow of a relaxing gas with internal degrees of freedom it is customary to use phenomenological equations for the macroscopic quantities, including the second viscosity, the heat conduction due to internal energy transport, and the internal energy itself. In a relaxing medium the internal energy is not a function of temperature, but depends on the process and is determined by the probabilities of energy transfer between molecules during collisions. For the internal energy of the gas it is usual to employ a so-called relaxation equation of the type

\[ \frac{de}{dt} = \epsilon - \epsilon^{(0)}(T) \]

\[ \tau(T, p) \]

where \( \frac{d}{dt} \) is the substantive derivative, \( \epsilon^{(0)}(T) \) is the value of the internal energy in the equilibrium state, and \( \tau \) is the relaxation time.

In reality, however, expressions of this type are valid only if very special assumptions are made concerning the nature of energy transfer between molecules. Moreover, in the general case it is usually impossible to write a single equation for the internal energy. The derivation of rigorous equations must be based on a kinetic analysis.

A number of authors have attempted to derive from the Boltzmann equation macroscopic equations describing the motion of a gas with the internal degrees of freedom taken into account. In [1] the problem was formulated and the limiting cases of very fast and very slow transitions — from translational energy to internal energy of the molecules — discussed. This treatment (unfortunately, known to the author only in the form of a very brief discussion of the results in [2]) has formed the basis of all subsequent attempts.

In [3] the influence of vibrational degrees of freedom on the coefficients of viscosity and heat conductivity was studied for the case where the time to establish equilibrium between vibrational and translational degrees of freedom is much less than the characteristic time of the process. In [4] Zhigulev considered the case of flows with vibrational relaxation on the assumption that the probability of vibrational energy transfer between molecules during collisions is of the same order as the probabilities of transfer of translational energy. In [5], which appeared after this paper had been prepared for the press, Vallander and Nagnibeda investigated the equations for a gas with internal degrees of freedom on the assumption that the gas is always in equilibrium with respect to translational velocities. The contents of [5] are closely related to §4 of this paper, the subject of which is the general case of a gas with an arbitrary number of relaxation times. It is shown, in particular, that in the general case the diagonal terms of the stress tensor contain, in addition to the pressure \( p = RT \), another term that is independent of the gradients. Then the temperature entering into the equation is not equal to the mean translational energy of the molecules.

1. Following [1], we shall consider the translational motions of the gas molecules in classical terms and the internal degrees of freedom in terms of quantum mechanics. Molecules in the quantum state \( \nu \) (with internal energy \( \epsilon^\nu \)) will be regarded as a gas of the \( \nu \)-th kind, so that the gas represents a mixture of gases differing in their quantum states. The state \( \nu \) may be defined by one or more quantum numbers characterizing the excitation of rotational, vibrational or electronic levels of the molecules.

Let \( f_\nu(t, x, \xi^\nu) \) be the velocity distribution function for molecules in the \( \nu \)-th state (of the \( \nu \)-th kind). Then Boltzmann's equation may be written in the form:

\[ \frac{df_\nu}{dt} = \frac{\partial f_\nu}{\partial \xi} + \xi \frac{\partial f_\nu}{\partial x} = \sum_{\mu kl} \int \left[ f_{\mu l} - f_{\mu l} \right] P_{\nu \mu}^{kl} (\xi^\nu, \xi^\mu, \xi^k, \xi^l) g_{\nu \mu} d\xi^k d\xi^l d\xi^\nu \]  

(1.1)

where \( P_{\nu \mu}^{kl} (\xi^\nu, \xi^\mu, \xi^k, \xi^l) \) is the probability (collision cross section) that as a result of a collision between molecules in states \( \nu \) and \( \mu \) with velocities \( \xi^\nu \) and \( \xi^\mu \), respectively, the molecule \( \nu \) goes over into the state \( k \) and acquires a velocity \( \xi^k \), while the \( \mu \)-molecule goes over into the state \( l \) and acquires a velocity \( \xi^l \). The distribution functions are normalized so that

\[ n^\nu (t, x) = \int f_\nu d\xi^\nu \]  

(1.2)
where \( n^\nu \) is the number of particles in state \( \nu \) in unit volume.

The mean velocity of the \( \nu \)-th component of the gas

\[
u^\nu = \frac{1}{n^\nu} \int \xi^\nu f d\xi^\nu .
\]

Since the masses of all the molecules are the same, the macroscopic velocity of the gas as a whole

\[
u = \frac{1}{n} \sum_{\nu} n^\nu \nu^\nu \quad \left( n = \sum_{\nu} n^\nu \right)
\]

where \( n \) is the total number of molecules in unit volume. Multiplying each of Eqs. (1.1) by \( \xi^\nu \) and \( (1/2)m \xi^\nu u^2 + E^\nu \) in turn, integrating with respect to \( \xi^\nu \), and summing the equations with respect to \( \nu \), we get the following conservation equations:

**Conservation of number of particles**

\[
\frac{\partial n}{\partial t} + \frac{\partial n u_r}{\partial x_r} = 0 .
\]

**Conservation of momentum**

\[
\left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right) u_t + \frac{1}{nm} \frac{\partial}{\partial x_i} P_{ij} = 0 .
\]

**Conservation of energy**

\[
n \left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right) \left( \frac{3}{2} kT + \epsilon \right) = - \frac{\partial q_r}{\partial x_r} - P_{tr} \frac{\partial u_t}{\partial x_r} .
\]

where

\[
P_{ij} = \sum_{\nu} m \int c_i c_j f d\xi^\nu \quad \left( c_i^\nu = \xi_i^\nu - u_i \right)
\]

\[
q_r = \sum_{\nu} \int c_i \left( \frac{m c_i^2}{2} + E^\nu \right) f d\xi^\nu
\]

\[
\frac{3}{2} kT = \frac{1}{n} \sum_{\nu} \frac{m c_i^2}{2} \int f d\xi^\nu
\]

\[
\epsilon = \frac{1}{n} \sum_{\nu} E^\nu n^\nu .
\]

The right sides of the equations vanish, since the total number of particles, their momentum and energy do not change during collisions. Thus, we have five equations with 14 unknowns: \( n, u_i, q_i, P_{ij} \), and \( \epsilon \) (the tensor \( P_{ij} \) is symmetrical and \( T \) can be expressed in terms of \( P_{ij} \)).

§2. Before proceeding to derive the missing relations, we shall describe a method, more convenient for our purpose, of expanding solutions of the Boltzmann equation with respect to a small parameter. Let us write Boltzmann's equation for a monatomic gas in the form:

\[
\frac{df}{dt} = \frac{1}{\epsilon} J = \frac{1}{\epsilon} \int \left( f(t', \xi') - f(t, \xi) \right) gP_{\xi \xi'}(\xi, \xi') \, d\xi d\xi' .
\]

where the same letters denote dimensionless functions referred to their characteristic values; then the parameter \( \epsilon \) is equal to the ratio of the mean time between collisions to the characteristic time of the process, i. e. , to the Knudsen number.

Integrating (2.1) over the trajectory of the particles with velocity \( \xi \), we have

\[
f(t, x_0 + \xi (t - t_0), \xi) - f(t_0, x_0, \xi) = \frac{1}{\epsilon} \int_{t_0}^{t} J(s, x_0 + \xi (s - t_0), \xi) \, ds .
\]

Let \( \tau = t - t_0 \) be of the order of \( \epsilon \). If over the length of the free path or in the time \( \tau = O(\epsilon) \) the distribution func-